Frequency-locking and heteroclinic cycles near subharmonic resonances in forced oscillators

Mohamed Belhaq
University Hassan II - Casablanca

Fifty Years of Finite Freedom Mechanics
Marseille, October 25-28, 2010

Outline

1. Introduction
   - Pendulum
   - Periodic response

Outline

Outline

1. Introduction
   - Pendulum
   - Periodic response
2. Effect of HFE on the dynamic of some forced oscillators
   - Forced and self-excited Duffing oscillator
   - Hertzian contact
   - Cantilever beam
   - Pull-in instability
3. Effect on heteroclinic cycle
   - 3:1 Heteroclinic cycle
   - 4:1 Heteroclinic cycle

Outline

Introduction

- **Faraday (1831):** Shown that particles on a vibrating surface move towards points where vibration are most intense,

Introduction

- **Faraday (1831):** Shown that particles on a vibrating surface move towards points where vibration are most intense,
- **Stephenson (1908):** Stabilization of an inverted pendulum using a fast vertical excitation,

Introduction

- **Faraday (1831):** Shown that particles on a vibrating surface move towards points where vibration are most intense,
- **Stephenson (1908):** Stabilization of an inverted pendulum using a fast vertical excitation,
- **Chelomei (1983):** Carried out experiments,

Introduction

Last decade: attention was focused on the effect of fast excitations on certain characteristics of mechanical systems, such as

- equilibrium stability (**Thomsen 2002**),

Introduction

Last decade: attention was focused on the effect of fast excitations on certain characteristics of mechanical systems, such as

- equilibrium stability (**Thomsen 2002**),
- natural frequency (**Jensen et al. 2000**),

Introduction

Last decade: attention was focused on the effect of fast excitations on certain characteristics of mechanical systems, such as

- equilibrium stability (**Thomsen 2002**),
- natural frequency (**Jensen et al. 2000**),
- buckling of elastic beams (**Jensen 2000**),

Introduction

Last decade: attention was focused on the effect of fast excitations on certain characteristics of mechanical systems, such as

- equilibrium stability (**Thomsen 2002**),
- natural frequency (**Jensen et al. 2000**),
- buckling of elastic beams (**Jensen 2000**),
- symmetry breaking (**Mann et Koplow 2006**),

Introduction

Last decade: attention was focused on the effect of fast excitations on certain characteristics of mechanical systems, such as

- equilibrium stability (**Thomsen 2002**),
- natural frequency (**Jensen et al. 2000**),
- buckling of elastic beams (**Jensen 2000**),
- symmetry breaking (**Mann et Koplow 2006**),
- limit cycle, saddle-node, hysteresis, pull-in instability in MEMS and heteroclinic cycles (**MB et *al*. 2007-2010**).

✓ **Objectives:**

- Show the effect of high-frequency excitation (HFE) on the dynamic of some mechanical systems,

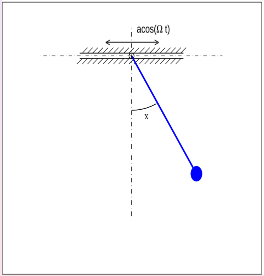
✓ **Objectives:**

- Show the effect of high-frequency excitation (HFE) on the dynamic of some mechanical systems,
- Study the influence of HFE on frequency-locking and heteroclinic cycles.

Outline

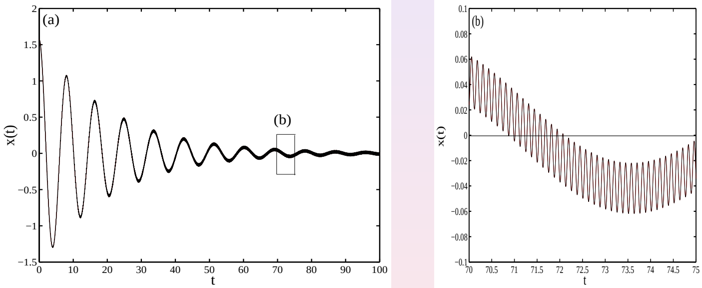
✓ Pendulum

Assume that the point of suspension of the pendulum is submitted to a horizontal HFE:



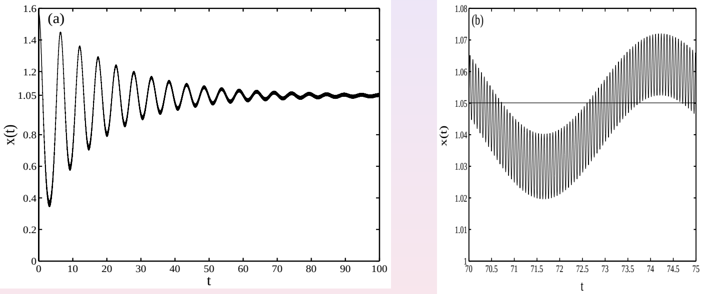
$$\ddot{x} + \alpha\dot{x} + \sin x = a\Omega^2 \cos x \cos \Omega t$$

✓ **Numerical simulation for $\Omega = 50$**



The point $x = 0$ is a quasi-static stable equilibrium.

✓ Numerical simulation for $\Omega = 100$



- The position $x = 0$ becomes unstable,
- A new quasi-static equilibrium $x \approx 1.05$ appears,
- Non-trivial effect: The frequency of the slow motion changes.

✓ **Analysis**

We use the method of separation on motion (Blekhman 2000) by introducing two time-scales:

$$T_0 = \Omega t, \quad T_1 = t$$

- Fast time T_0 for the motion at the rate of Ω.

- Slow time T_1 for the motion at the rate of the natural frequency.

- We split motion into a slow part and a fast part:

$$x(t) = z(T_1) + \epsilon\varphi(T_0, T_1) \tag{1}$$

✓ **Result**

- Equation of the low dynamic

$$\ddot{z} + \alpha \dot{z} + \left(1 - \frac{1}{2}(a\Omega)^2 \cos z\right) \sin z = 0$$

- Stability analysis of equilibria indicates that:

Pendulum

- The stable position $z = 0$ becomes unstable for a certain value of the frequency,

Pendulum

- The stable position $z = 0$ becomes unstable for a certain value of the frequency,
- New stable positions appear: $z = \pm \arccos(2/(a\Omega)^2)$,

Pendulum

- The stable position $z = 0$ becomes unstable for a certain value of the frequency,
- New stable positions appear: $z = \pm \arccos(2/(a\Omega)^2)$,
- The position $\bar{z} = \pi$ cannot be stabilized by a horizontal HFE,

Pendulum

- The stable position $z = 0$ becomes unstable for a certain value of the frequency,
- New stable positions appear: $z = \pm \arccos(2/(a\Omega)^2)$,
- The position $\bar{z} = \pi$ cannot be stabilized by a horizontal HFE,
- Instead, a vertical HFE can stabilize this position (**Stephenson 1908**).

Outline

Effect of horizontal FHE on limit cycle

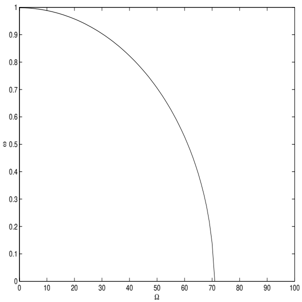
Consider a self-excited oscillator subjected to a horizontal FHE

$$\frac{d^2x}{dt^2} + (-\alpha + \beta x^2)\frac{dx}{dt} + \sin x = a\Omega^2 \cos x \cos \Omega t \qquad (2)$$

Averaging and using perturbation analysis leads to a relationship between the frequency of the limit cycle ω and Ω:

$$\omega = \omega_0 + \epsilon^2\left(\frac{-3(\delta - (\gamma a\Omega)^2)A^2}{8\omega_0}\right) + ... \qquad (3)$$

$$\omega = \omega_0 + \epsilon^2 \left(\frac{-3(\delta - (\gamma a \Omega)^2) A^2}{8\omega_0} \right) + \ldots \tag{4}$$



$$72 < \Omega_{cr} < 73$$

Figure: Effect of horizontal fast excitation on limit cycle

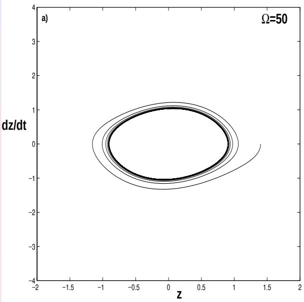
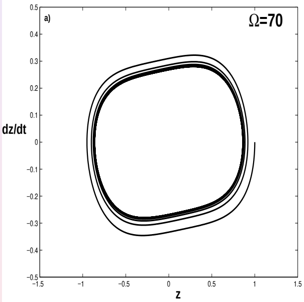


Figure: Effect of horizontal fast excitation on limit cycle



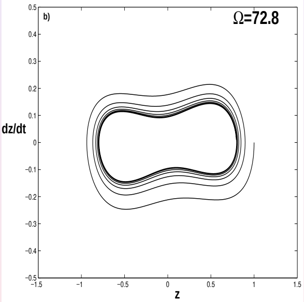
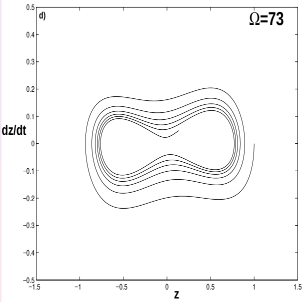


Figure: Effect of horizontal fast excitation on limit cycle

Figure: Effect of horizontal fast excitation on limit cycle



Effect of vertical FHE on limit cycle

$$\frac{d^2x}{dt^2} + (-\alpha + \beta x^2)\frac{dx}{dt} + \sin x = a\Omega^2 \sin x \cos \Omega t \qquad (5)$$

Figure: Effect of vertical fast excitation on limit cycle

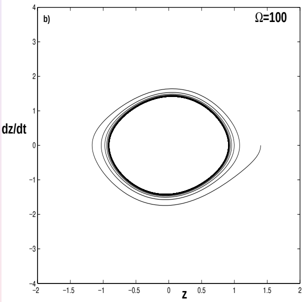


Figure: Effect of vertical fast excitation on limit cycle

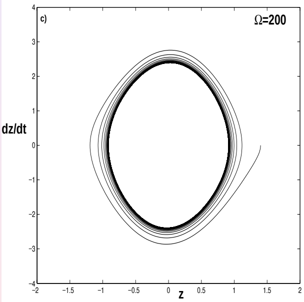
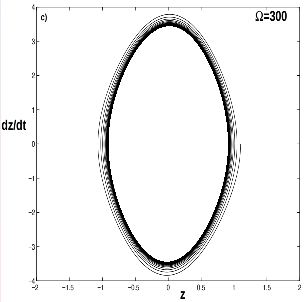


Figure: Effect of vertical fast excitation on limit cycle



UNIVERSITE HASSAN II-AIN CHOCK

Outline

جامعة الحسن الثاني . عين الشق
UNIVERSITE HASSAN II-AIN CHOCK

Consider the forced van der Pol-Duffing oscillator under horizontal HFE:

$$\ddot{x} + x - (\alpha - \beta x^2)\dot{x} - \gamma x^3 = h \cos \omega t + a\Omega^2 \cos x \cos \Omega t \quad (6)$$

Averaging, the equation of the slow dynamic reads :

$$\ddot{z} + \omega_0^2 z - (\alpha - \beta z^2)\dot{z} - \xi z^3 = h \cos \omega t \tag{7}$$

with

$$\omega_0^2 = 1 - \frac{1}{2}(a\Omega)^2$$

$$\xi = \gamma - \frac{1}{3}(a\Omega)^2$$

Dynamic near the fundamental resonance 1:1:

Near the fundamental resonance 1:1, the modulation equations are

$$\begin{cases} \frac{dr}{dt} = & Ar - Br^3 - H\sin(\theta) \\ r\frac{d\theta}{dt} = & Sr - Cr^3 - H\cos(\theta) \end{cases} \tag{8}$$

Analysis of equilibria leads to the frequency response

✓ Resonance curves near the principal resonance

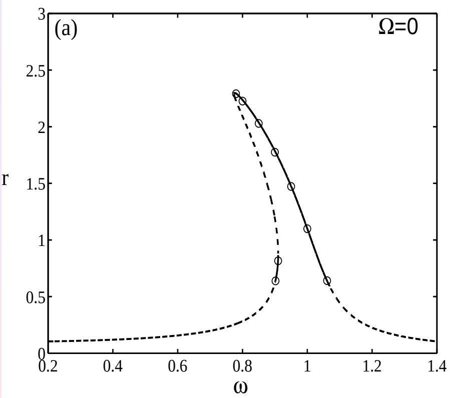


Figure: Frequency response for the principal resonance 1:1

✓ Resonance curves near the principal resonance

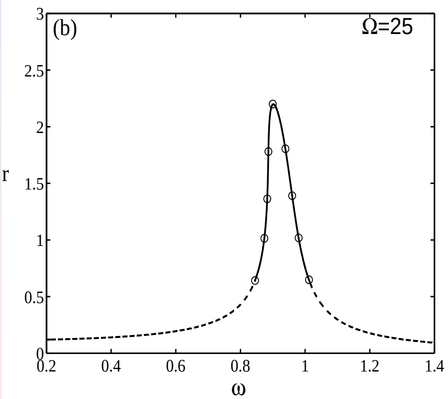


Figure: Frequency response for the principal resonance 1:1

✓ Resonance curves near the principal resonance

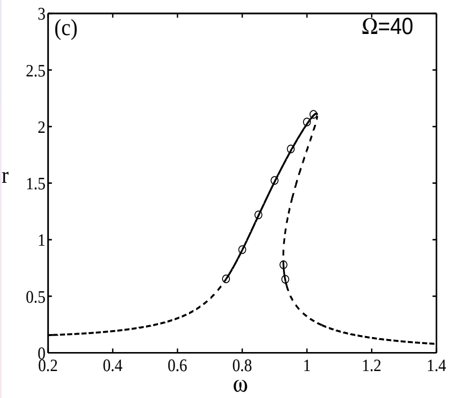


Figure: Frequency response for the principal resonance 1:1

✓ **Quasiperiodic (QP) response and frequency-locking**

Periodic solution of the modulation equations:

$$\begin{cases} \frac{dr}{dt} = & Ar - Br^3 - H\sin(\theta) \\ r\frac{d\theta}{dt} = & Sr - Cr^3 - H\cos(\theta) \end{cases} \quad Weapplyaperturbationmethodtoappr$$

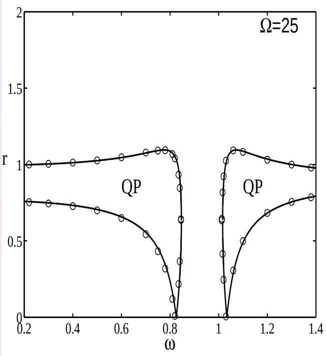
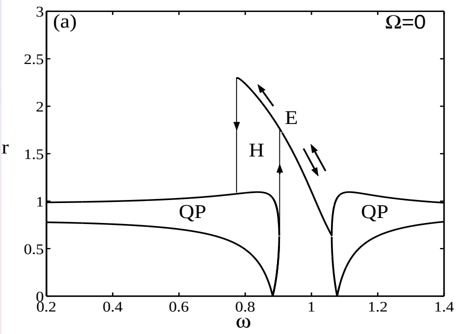
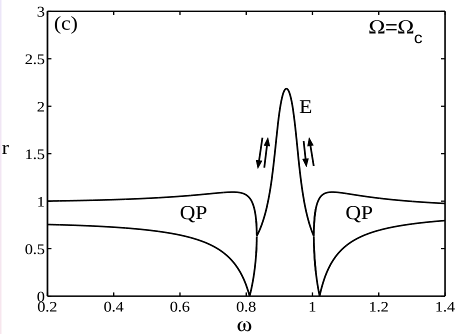
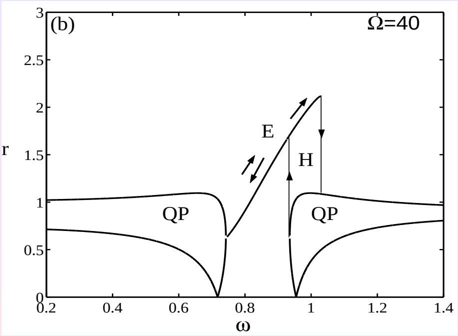


Figure: QP modulation domain


(a)
Ω=0
r
ω
E
H
QP
QP
0
0.5
1
1.5
2
2.5
3
0.2
0.4
0.6
0.8
1
1.2
1.4


(c)
Ω=Ωc
E
QP
QP
r
ω
0
0.5
1
1.5
2
2.5
3
0.2
0.4
0.6
0.8
1
1.2
1.4


(b)
Ω=40
r
ω
E
H
QP
QP
0
0.5
1
1.5
2
2.5
3
0.2
0.4
0.6
0.8
1
1.2
1.4

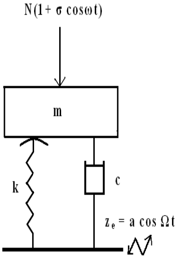
Outline

✓ Control of vibro-impact dynamics of a single-sided Hertzian contact forced oscillator:

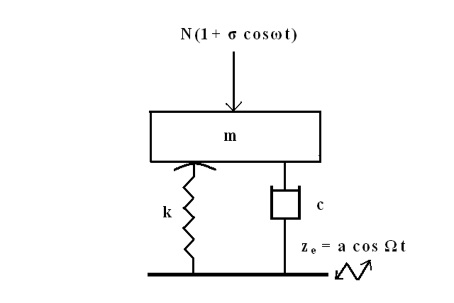
Consider a single-sided Hertzian contact forced oscillator

Test rig consists of a steel ball preloaded between two horizontal steel flat surfaces (Perret-Liaudet 2006). The first one is fixed to a heavy rigid frame and the second one is rigidly fixed to a vertically moving cylinder.

Fig. 11. Photo of the used dynamic test rig.



✓ Case of a fast harmonic base motion:



Effect of a fast harmonic base motion

$$m\ddot{z} + c(\dot{z} - \dot{z}_e) + k(z - z_e)^{\frac{3}{2}} = N(1 + \sigma \cos \omega t) \qquad (10)$$

z: normal displacement of the rigid mass m,
c: damping coefficient,
k: constant given by the Hertzian theory,
N: static normal force,
σ and ω are the level of the excitation and its frequency.
$z_e = a \cos \Omega t$ the fast harmonic base motion.

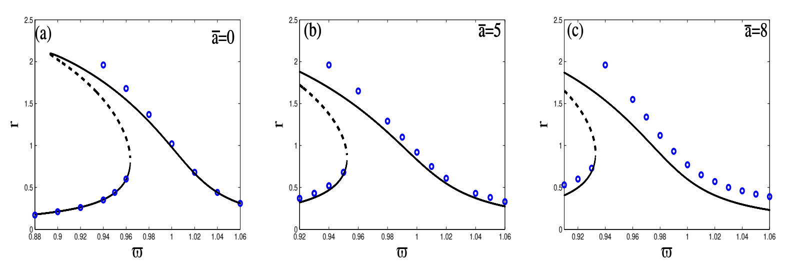
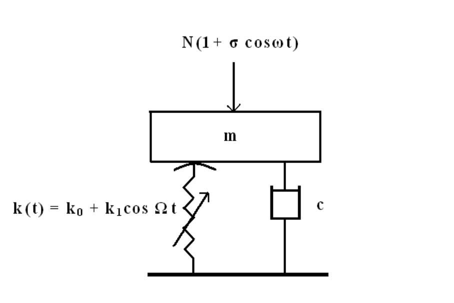


Figure: Amplitude frequency response shifts **left**

✓ Case of a fast parametric stiffness:



✓ **The model:**

$$m\ddot{z} + c\dot{z} + (k_0 + k_1 \cos \Omega t) z^{\frac{3}{2}} = N(1 + \sigma \cos(\omega t)) \qquad (11)$$

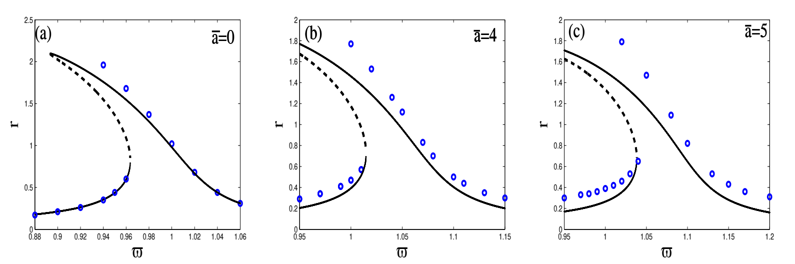


Figure: Amplitude frequency response shifts **right**

✓ **Control of vibro-impact dynamics of a Hertzian contact forced oscillator using electromagnetic force:**

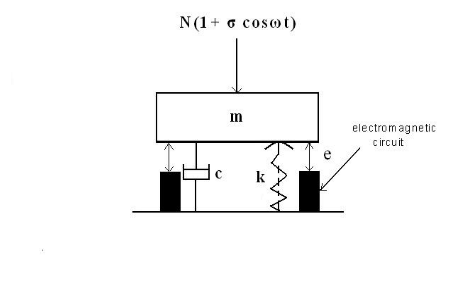


Figure: Dynamic model with electromagnetic force

$$m\ddot{\delta} + c\dot{\delta} + k\delta^{3/2} = N(1 + \sigma\cos\omega t) + \frac{C_0 I^2}{(e-\delta)^2}$$

For fixed parameters and $\Omega = 8$:

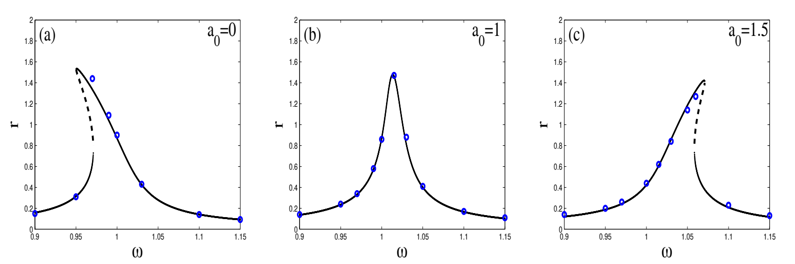
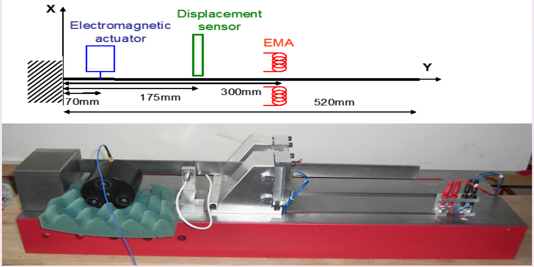


Figure: Frequency response shifts **right** and bends from softening to hardening

Outline

✓ **Control of THE dynamic of a cantilever beam using electromagnetic force:**

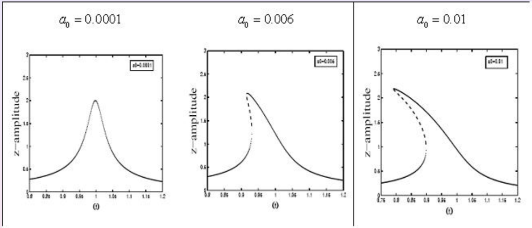
The test is composed of a clamped-free flexible steel beam

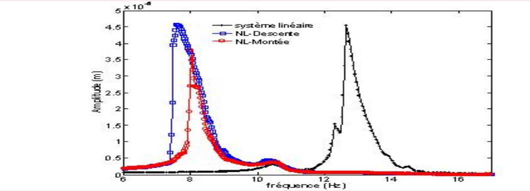


Equation of motion

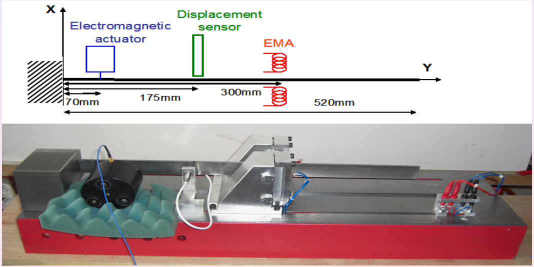
$$m\ddot{\delta} + \alpha\dot{\delta} + k\delta = F\cos\omega t + Cl^2\left(\frac{1}{(\lambda-\delta)^2} - \frac{1}{(\lambda+\delta)^2}\right) \qquad (12)$$

Using DC


$a_0 = 0.0001$
$a_0 = 0.006$
$a_0 = 0.01$
z−amplitude
ω


système linéaire
NL-Descente
NL-Montée
Amplitude (m)
fréquence (Hz)

Using AC with high-frequency



For fixed amplitude of the AC

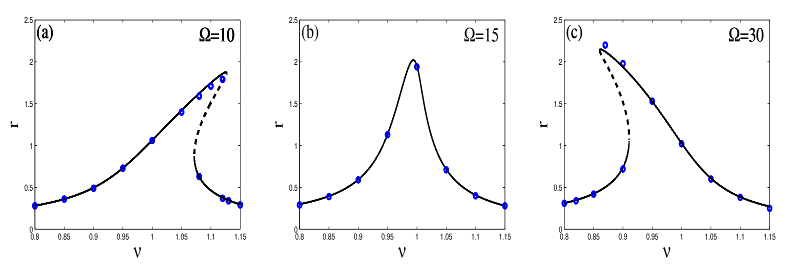
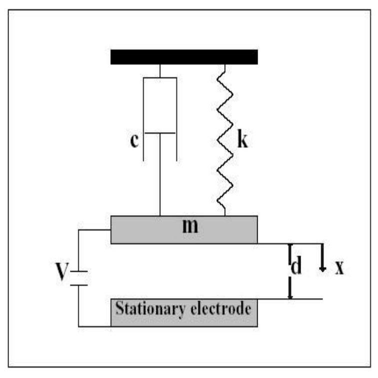


Figure: Frequency response shifts **left** and bends from hardening to softening

Outline

✓ Control of pull-in using HF actuation

A single-degree-of-freedom model representing a capacitive MEMS device employing a DC and AC voltages as actuator.



ù

Figure: Effect HFV on static pull-in

The model:

$$m\ddot{x} + c\dot{x} + kx = \frac{\varepsilon A}{2(d-x)^2} V^2 \tag{13}$$

ε: dielectric constant of the gap medium,
d is the initial capacitor gap width,
A is the area of the cross section,
and V is the electric load.

With

$$V = V_0 + V_1 + Ucos(\Omega^* t)$$

the dimensionless equation is given by

$$X'' + 2\xi X' + X = \frac{\alpha}{(1-X)^2} + \frac{\beta \cos(\omega\tau)}{(1-X)^2} + \frac{\gamma \cos(2\omega\tau)}{(1-X)^2} \quad (14)$$

Effect on Static pull-in

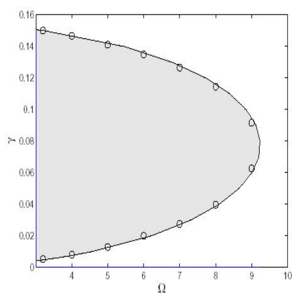


Figure: Domain of pull-in suppression in the plane amplitude of the HFV vs the frequency

Outline

✓ 3:1 Heteroclinic cycle

- Approximation of 3:1 heteroclinic cycle

UNIVERSITE HASSAN II-AIN CHOCK

✓ 3:1 Heteroclinic cycle

- Approximation of 3:1 heteroclinic cycle
- Forced van der Pol-Duffing oscillator:

$$\ddot{x} + \omega_0^2 x - (\alpha - \beta x^2)\dot{x} - \gamma x^3 = h \cos \omega t \tag{15}$$

✓ 3:1 Heteroclinic cycle

- Approximation of 3:1 heteroclinic cycle
- Forced van der Pol-Duffing oscillator:

$$\ddot{x} + \omega_0^2 x - (\alpha - \beta x^2)\dot{x} - \gamma x^3 = h \cos \omega t \tag{15}$$

- 3:1 resonance condition,

$$\omega_0^2 = (\frac{\omega}{3})^2 + \sigma \tag{16}$$

✓ 3:1 Heteroclinic cycle

- Approximation of 3:1 heteroclinic cycle
- Forced van der Pol-Duffing oscillator:

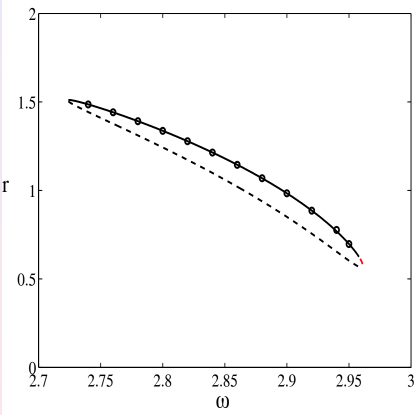
$$\ddot{x} + \omega_0^2 x - (\alpha - \beta x^2)\dot{x} - \gamma x^3 = h \cos \omega t \tag{15}$$

- 3:1 resonance condition,

$$\omega_0^2 = (\frac{\omega}{3})^2 + \sigma \tag{16}$$

- Slow flow system

Amplitude-frequency response near 3:1 resonance



✓ **Quasiperiodic modulation region**

Perturbation analysis on the slow flow system gives the **slow slow** flow

$$\frac{dR}{dt} = AR - BR^3 \quad (17)$$

$$\frac{d\varphi}{dt} = -CR^2 \quad (18)$$

with

The envelope of the modulated amplitude of QP response

$$r_{min} = \sqrt{R^2 + \frac{R^4}{9S^2}(H_1^2 + H_2^2) - \frac{2R^3}{3S}\sqrt{H_1^2 + H_2^2}} \tag{19}$$

$$r_{max} = \sqrt{R^2 + \frac{R^4}{9S^2}(H_1^2 + H_2^2) + \frac{2R^3}{3S}\sqrt{H_1^2 + H_2^2}} \tag{20}$$

جامعة الحسن الثاني . عين الشق
UNIVERSITE HASSAN II-AIN CHOCK

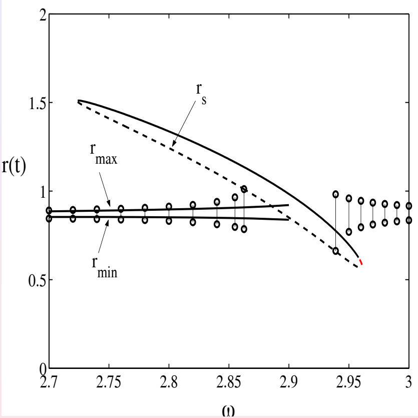


Figure: Lines for analytic and circles for numeric.

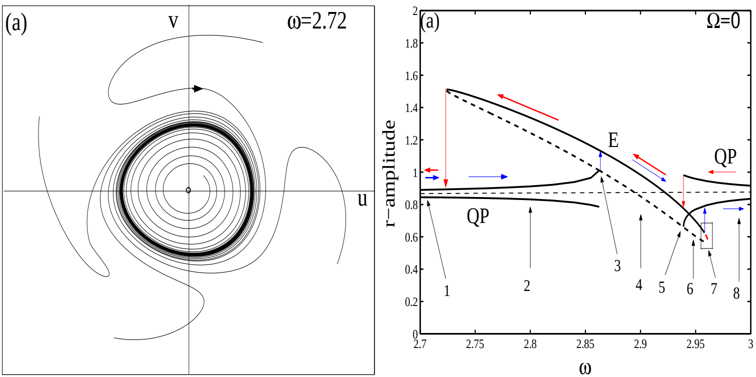


Figure: Phase portrait 1

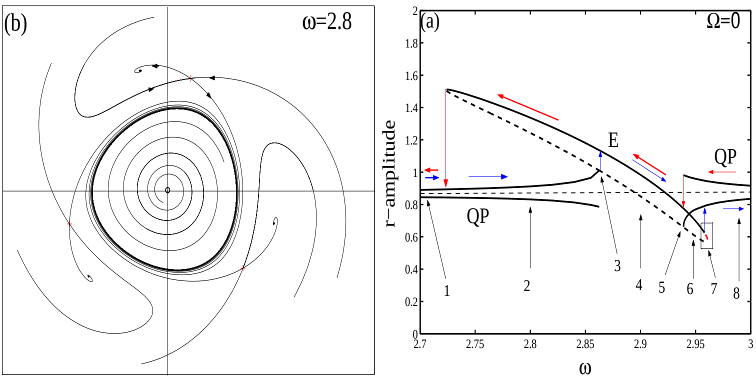


Figure: Phase portrait 2

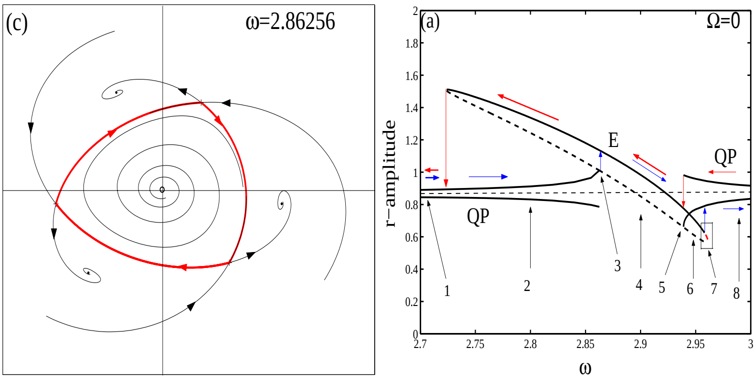


Figure: Phase portrait 3

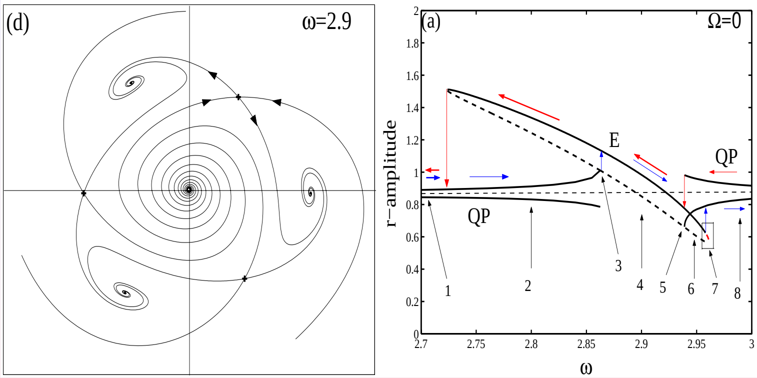


Figure: Phase portrait 4

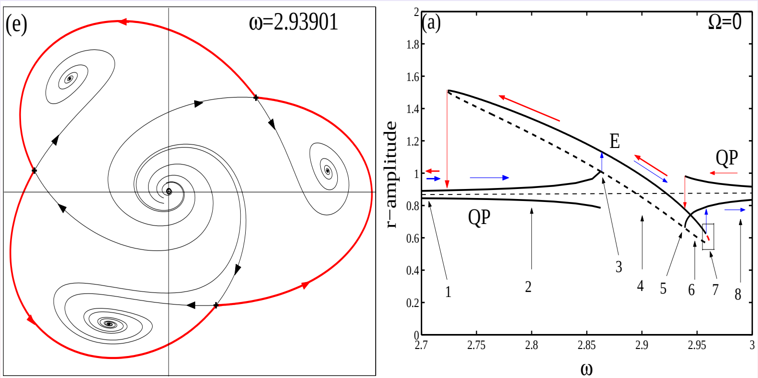


Figure: Phase portrait 5

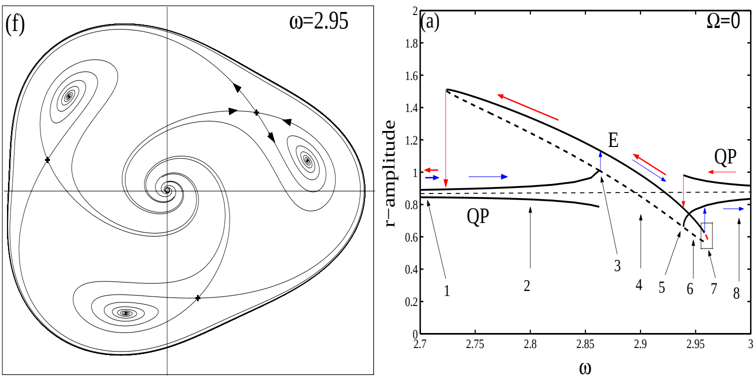
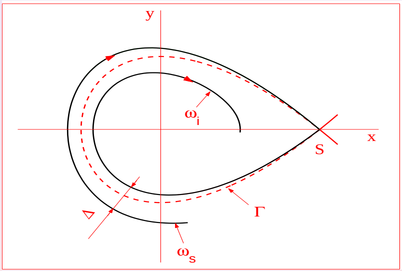


Figure: Phase portrait 6

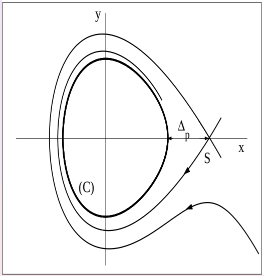
- **Melnikov** criterion:

- **Melnikov** criterion:
- Vanishing distance between stable and unstable manifolds: $\Delta = 0$



✓ **Collision criterion**

Collision between limit cycle and saddle: $\Delta_p = 0$



- This criterion is equivalent to the Melnikov technique **[MB, Fiedler, Lakrad, 2000]**

- Collision criterion

$$r_{max} = r_s \tag{21}$$

- Collision criterion

$$r_{max} = r_s \quad (21)$$

- r_{max} is the upper limit of the limit cycle amplitude.

- Collision criterion

$$r_{max} = r_s \tag{21}$$

- r_{max} is the upper limit of the limit cycle amplitude.
- r_s is the amplitude of the saddles.

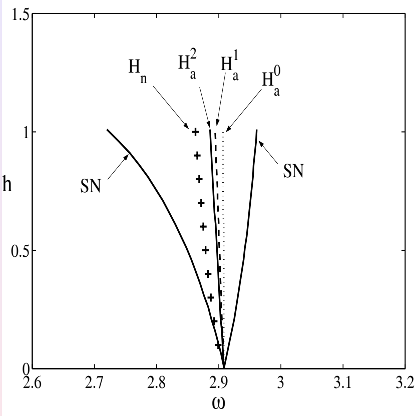


Figure: Heteroclinic bifurcation H_n for numeric, H_a for analytic

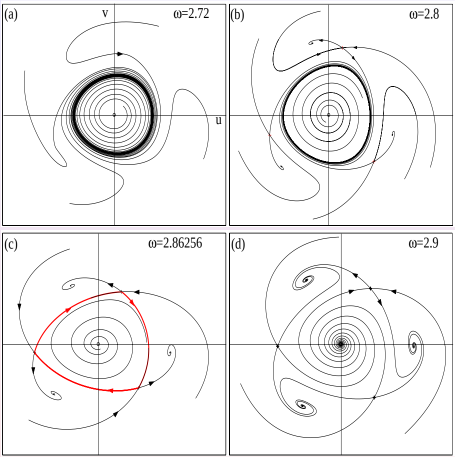


Figure: Analytical approximation of this inner heteroclinic bifurcation

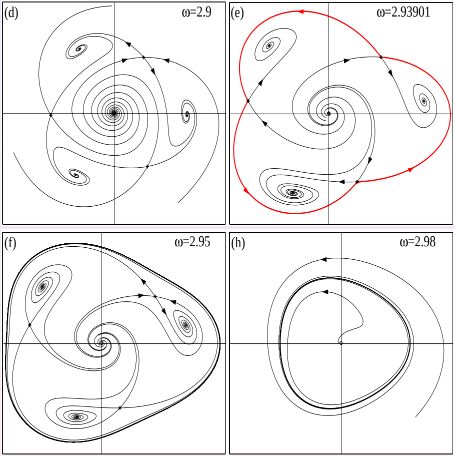


Figure: No analytical approximation of this outer heteroclinic bifurcation

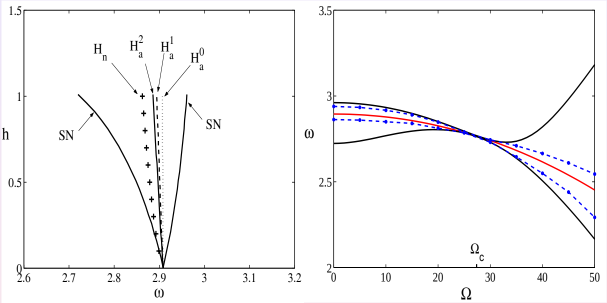


Figure: Saddle-node (black), heteroclinic connection, analytical (red) and numerical (bleu).

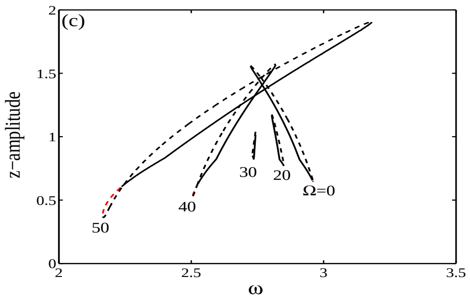


Figure: Effect of the frequency Ω on the resonance 3:1

Outline

✓ 4:1 Heteroclinic cycle

- Forced van der Pol-Duffing oscillator:

$$\ddot{x} + \omega_0^2(1 + h\cos\omega t)x - (\alpha - \beta x)\dot{x} - cx^2 = 0 \qquad (22)$$

✓ 4:1 Heteroclinic cycle

- Forced van der Pol-Duffing oscillator:

$$\ddot{x} + \omega_0^2(1 + h\cos\omega t)x - (\alpha - \beta x)\dot{x} - cx^2 = 0 \qquad (22)$$

- 4:1 resonance condition:

$$\omega_0^2 = (\frac{\omega}{4})^2 + \sigma \qquad (23)$$

✓ 4:1 Heteroclinic cycle

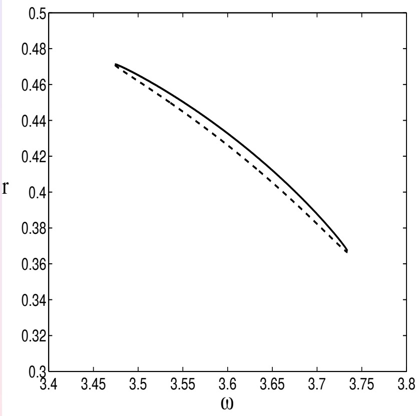
- Forced van der Pol-Duffing oscillator:

$$\ddot{x} + \omega_0^2(1 + h\cos\omega t)x - (\alpha - \beta x)\dot{x} - cx^2 = 0 \tag{22}$$

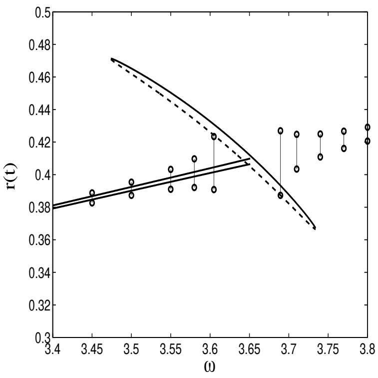
- 4:1 resonance condition:

$$\omega_0^2 = (\frac{\omega}{4})^2 + \sigma \tag{23}$$

- First perturbation gives the slow flow


0.5
0.48
0.46
0.44
0.42
0.4
0.38
0.36
0.34
0.32
0.3
r
3.4
3.45
3.5
3.55
3.6
3.65
3.7
3.75
3.8
ω

Second perturbation method on slow flow gives the **QP modulation domain**



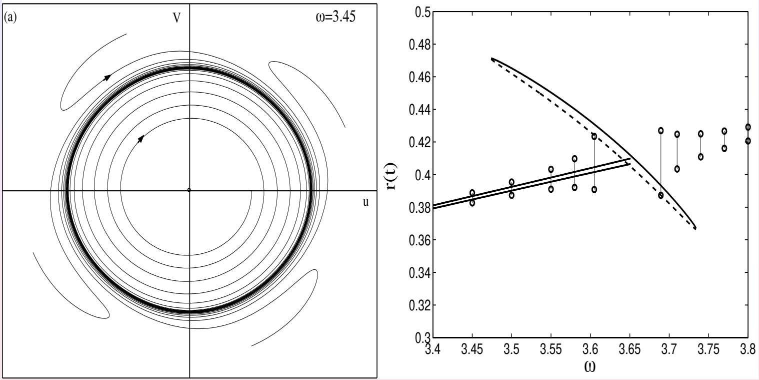


Figure: Phase portrait 1

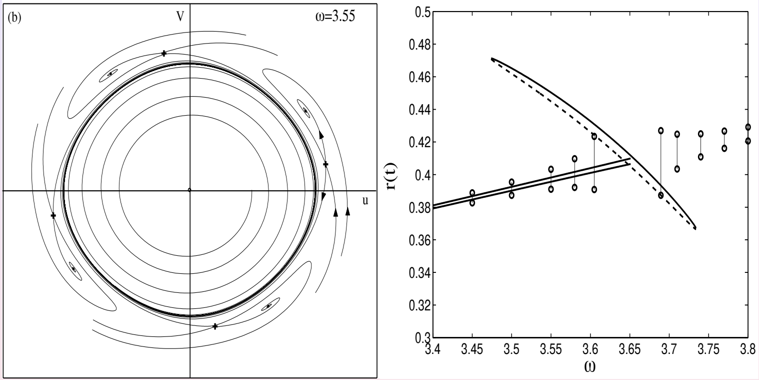


Figure: Phase portrait 2

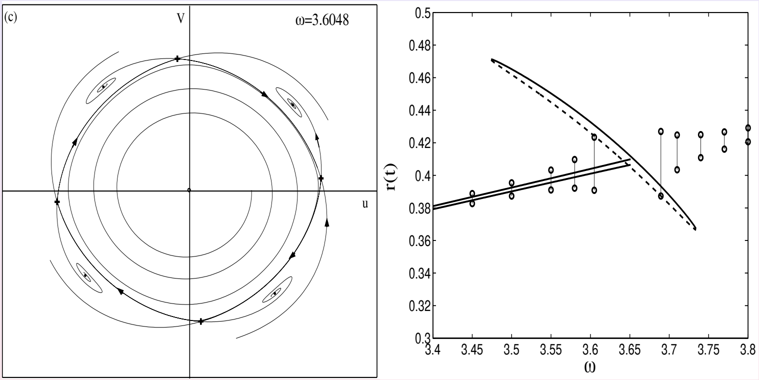


Figure: Phase portrait 3

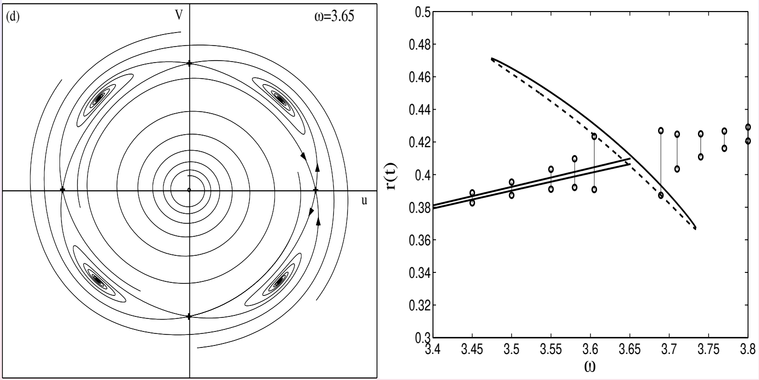
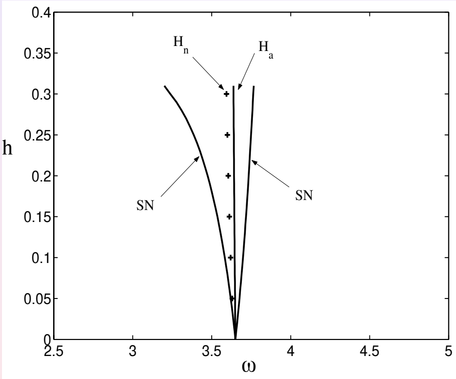
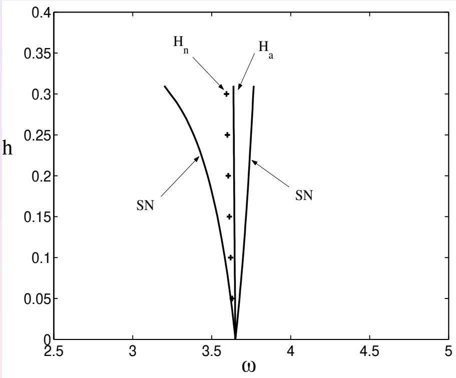


Figure: Phase portrait 4

- ✓ **Analytical approximation H_a (MB and Fahsi, 2010),**

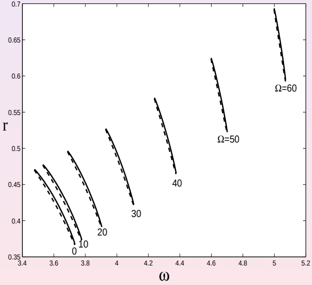


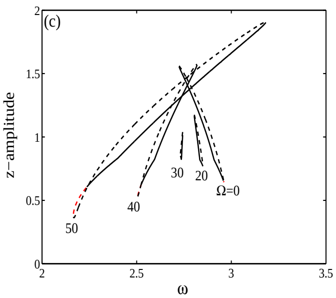
- ✓ **Analytical approximation H_a (MB and Fahsi, 2010),**



- ✓ **Numerical approximation H_n (MB, 1986).**

✓ Effect of High-frequency excitation on 3:1 and 4:1 resonances





✓ **Conclusion**:

- Bifurcation of heteroclinic cycles for 3:1 and 4:1 resonances **can be captured analytically** using the collision criterion,

✓ **Conclusion**:

- Bifurcation of heteroclinic cycles for 3:1 and 4:1 resonances **can be captured analytically** using the collision criterion,
- The 3:1 resonance **can be suppressed** in certain range of the HFE and its nonlinear characteristic can change,

✓ **Conclusion**:

- Bifurcation of heteroclinic cycles for 3:1 and 4:1 resonances **can be captured analytically** using the collision criterion,
- The 3:1 resonance **can be suppressed** in certain range of the HFE and its nonlinear characteristic can change,
- The 4:1 resonance **cannot be suppressed** by HFE, but undergoes a shift and its amplitude increases.

MERCI POUR VOTRE ATTENTION!

Some recent publications on the topic:

- M. Belhaq, A. Fahsi, Nonlinear Dyn (2009),
- A. Fahsi, M. Belhaq, Chaos, Solitons, Fractals (2009),
- F. Lakrad, M. Belhaq, Comm Nonlin Sc Num Simul (2010),
- A. Bichri, M. Belhaq, J. Perret-Liaudet, Nonlinear Dyn (2010),
- M. Belhaq, A. Fahsi, Nonlinear Dyn (2010),
- F. Lakrad, M. Belhaq, Int J Non-linear Mechanic (2010).
- M. Belhaq, A. Bichri, Der Hegopian, J. Mahfoud, Méca Industrie (2010).