Non-linear random vibrations: an amplitude-phase formulation

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With the complicity of Sergio Bellizzi, Djaffar Boussaa, Mario Defilippi...

This conference deals with Stochastic Nonlinear Dynamics to engineering problems.

The primary motivation is to approach the spectral responses of vibrating systems under wide-band random inputs. Non-linearities are strong, often localized, possibly including hysteresis.

Based on nonlinear modal analysis and the use of the Stochastic Averaging Principle, the concept of Equivalent Linear System with random parameters will be presented. The method is well suited for the characterization of nonlinear signatures, predictives responses, parametric identification and model reduction.

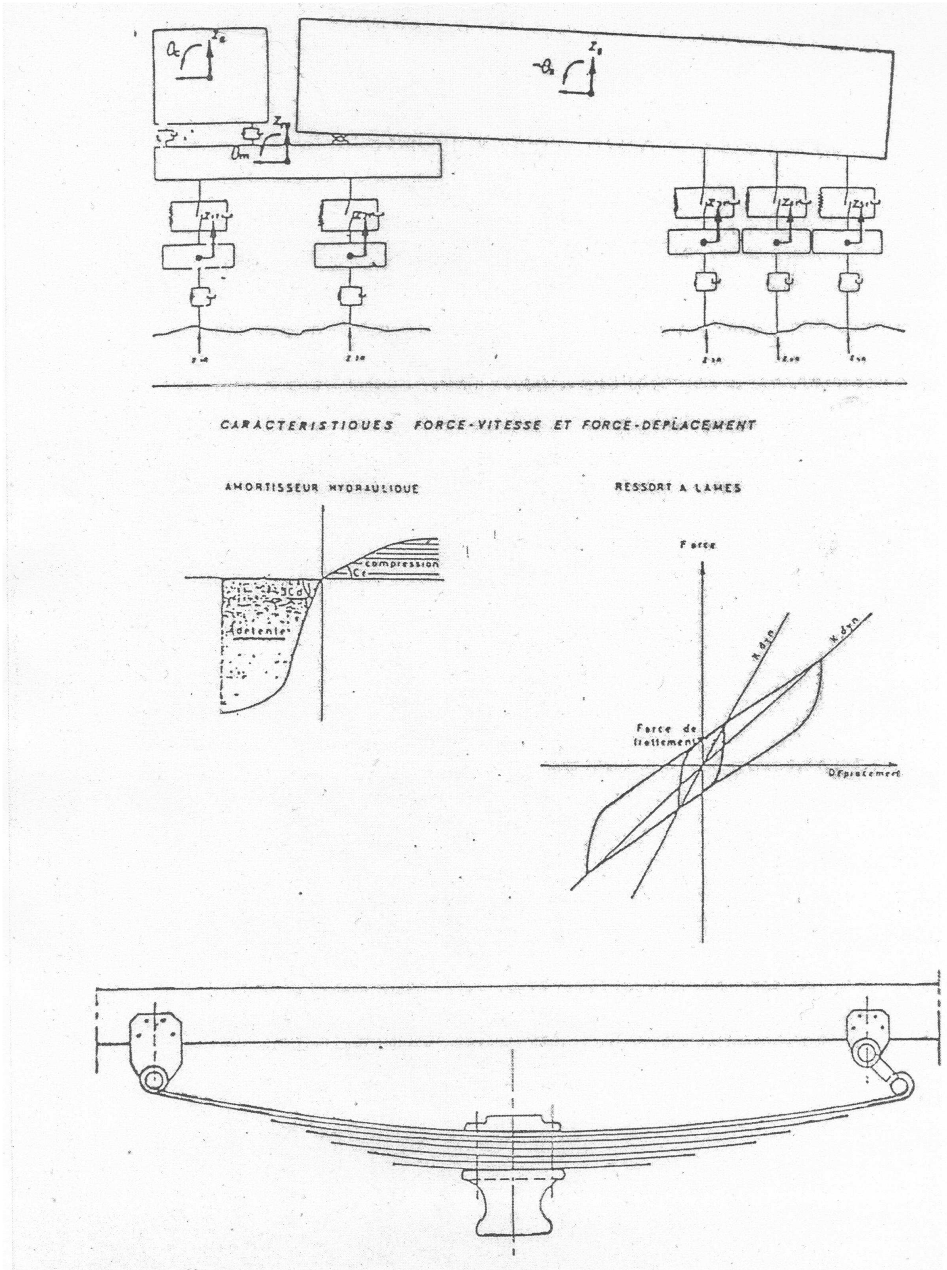

Figure 1. 10 DOF Truck

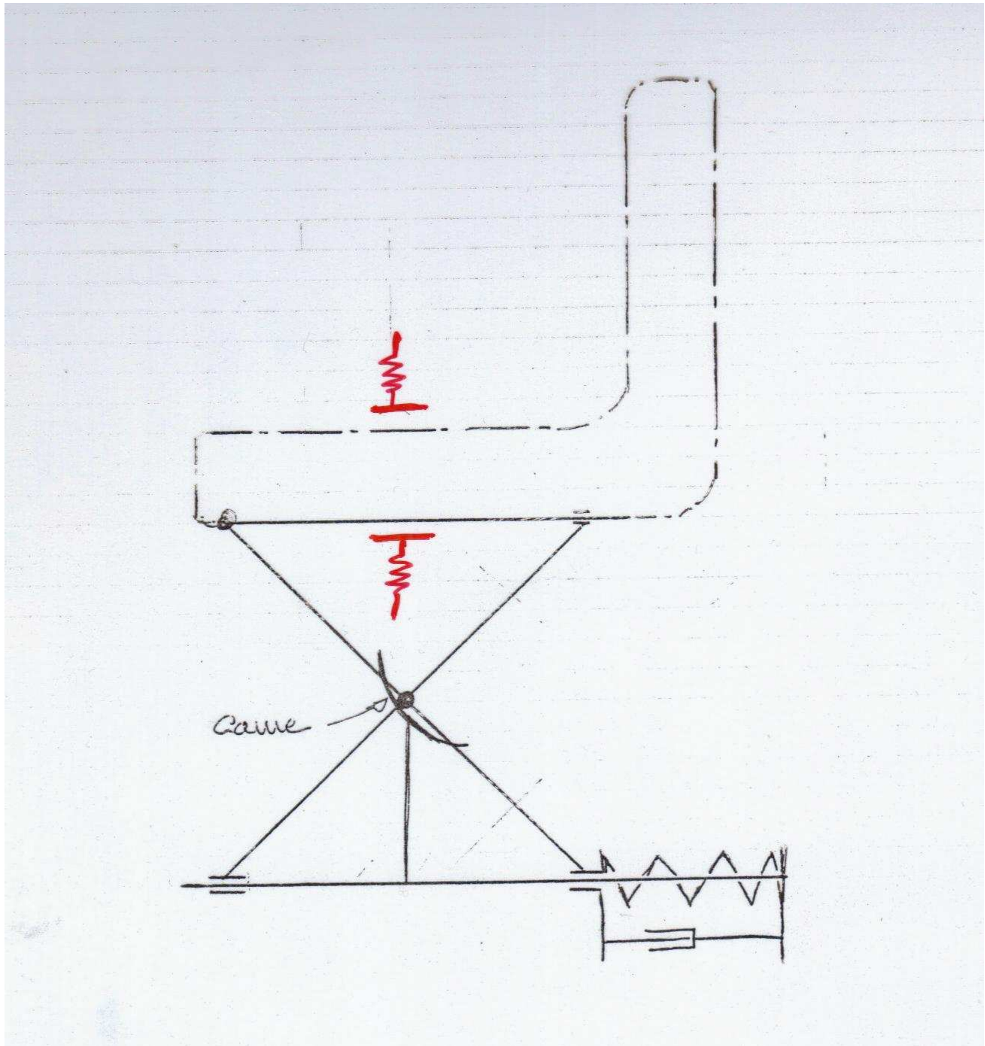

Figure 2. Seat: dry-friction and stops-system

GOVERNING EQUATION

$$[\boldsymbol{M}]\ddot{\boldsymbol{Q}}(t) + [\boldsymbol{C}(\dot{\boldsymbol{Q}}(t))]\dot{\boldsymbol{Q}}(t) + \boldsymbol{F}(\boldsymbol{Q}(t)) + [\boldsymbol{P}]\boldsymbol{\mathcal{H}}(t) = [\boldsymbol{S}]\boldsymbol{W}(t)$$

$$\mathcal{H}_i = k_{si}\, x_i + z_i \,, \quad \dot{z}_i + \beta_i\, |\dot{x}_i| z_i + \gamma_i\, \dot{x}_i\, |z_i| = (k_i - k_{si})\, \dot{x}_i \,, \quad \beta_i > 0$$

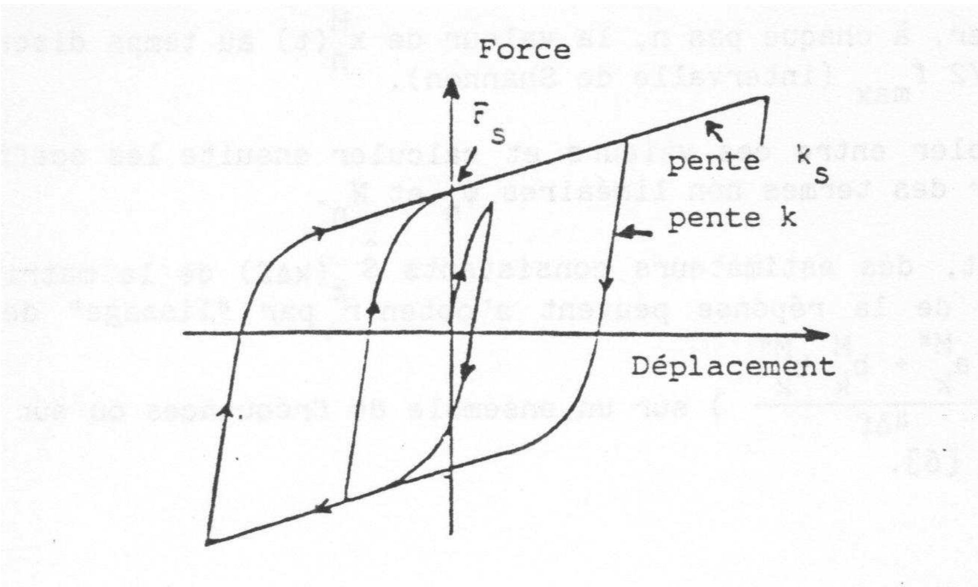

Figure 3. Model β=γ

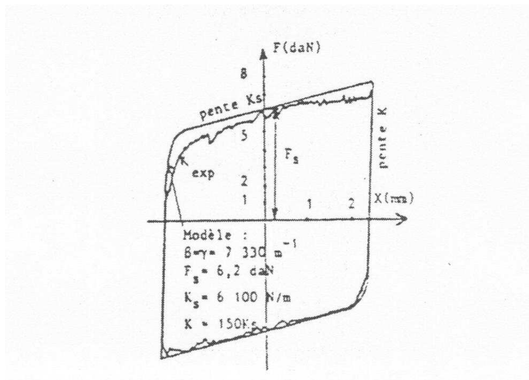

Figure 4. Model+experiment

LINEARIZATION WITH CONSTANT PARAMETERS

$$\boldsymbol{G}(\ddot{\boldsymbol{Q}}\,,\dot{\boldsymbol{Q}}\,,\boldsymbol{Q}) = \boldsymbol{W} \qquad (\text{st. pr. } \mathbb{E}(\boldsymbol{W}) = 0)$$

$$[\boldsymbol{M}]^{eq}\ddot{\boldsymbol{Q}} + [\boldsymbol{C}]^{eq}\dot{\boldsymbol{Q}} + [\boldsymbol{K}]^{eq}\boldsymbol{Q} + \boldsymbol{B}^{eq} = \boldsymbol{W}$$

$$\boldsymbol{e} = \boldsymbol{G}(\ddot{\boldsymbol{Q}}\,,\dot{\boldsymbol{Q}}\,,\boldsymbol{Q}) - ([\boldsymbol{M}]^{eq}\ddot{\boldsymbol{Q}} + [\boldsymbol{C}]^{eq}\dot{\boldsymbol{Q}} + [\boldsymbol{K}]^{eq}\boldsymbol{Q} + \boldsymbol{B}^{eq})$$

$$\min_{[\boldsymbol{M}]^{eq},\ldots\ldots,[\boldsymbol{B}]^{eq}} = \mathbb{E}(\boldsymbol{e}^{\boldsymbol{T}}\boldsymbol{e})$$

- Probability law of $(\ddot{\boldsymbol{Q}}\,,\dot{\boldsymbol{Q}}\,,\boldsymbol{Q})$ is in general unknown to perform the expectation $\mathbb{E}(.)$.

GAUSSIAN CLOSURE

Gaussian property: $\boldsymbol{Y} = \tilde{\boldsymbol{Y}} + \bar{\boldsymbol{Y}}$, $[\mathrm{cov}(\boldsymbol{Y})] = \mathbb{E}[\tilde{\boldsymbol{Y}}\tilde{\boldsymbol{Y}}^{\boldsymbol{T}}]$

$$\mathbb{E}\Big[h(\boldsymbol{Y})\tilde{\boldsymbol{Y}}^{T}\Big] = (\mathbb{E}[\,\partial_{\boldsymbol{Y}} h\,])\mathbb{E}[\tilde{\boldsymbol{Y}}\tilde{\boldsymbol{Y}}^{\boldsymbol{T}}] \text{ for some scalar } h(\boldsymbol{Y})$$

Let $\boldsymbol{Y} = \mathrm{col}(\ddot{\boldsymbol{Q}}, \dot{\boldsymbol{Q}}, \boldsymbol{Q})$, it follows

$$\boldsymbol{B}^{eq} + [\boldsymbol{K}]^{eq}\,\mathbb{E}(\boldsymbol{Q}) = 0$$

$$[\boldsymbol{M}]^{eq} = \mathbb{E}\Big[\partial_{\ddot{\boldsymbol{Q}}}\boldsymbol{G}\Big],\ [\boldsymbol{C}]^{eq} = \mathbb{E}\Big[\partial_{\dot{\boldsymbol{Q}}}\boldsymbol{G}\Big],\ [\boldsymbol{K}]^{eq} = \mathbb{E}[\partial_{\boldsymbol{Q}}\boldsymbol{G}]$$

- Depend on $(\bar{\boldsymbol{Y}}, [\mathrm{cov}(\boldsymbol{Y})])$ (iterative process).

” TRUE ” STOCHASTIC LINEARIZATION METHOD

- With signal processing: find $([\boldsymbol{M}]^{eq}, [\boldsymbol{C}]^{eq}, [\boldsymbol{K}]^{eq}, \boldsymbol{B}^{eq})$ s.t.

$$\lim_{T\to\infty} \frac{1}{T} \int_T (\boldsymbol{e}^{\boldsymbol{T}} \boldsymbol{e}) dt \quad \text{is minimized}$$

$$\boldsymbol{e} = \boldsymbol{W} - ([\boldsymbol{M}]^{eq} \ddot{\boldsymbol{Q}} + [\boldsymbol{C}]^{eq} \dot{\boldsymbol{Q}} + [\boldsymbol{K}]^{eq} \boldsymbol{Q} + \boldsymbol{B}^{eq})$$

and $(\boldsymbol{W}, \ddot{\boldsymbol{Q}}, \dot{\boldsymbol{Q}}, \boldsymbol{Q})$ are measured values on $(0,T)$.

- Do not use the mathematical form of the non-linear elements

COMMENT

- Good approximation of the first two moments: mean value and covariance matrix (energy), even for large nonlinearities

- Very bad approximation of two-time statistics such that Autocorrelation Matrix or Power Spectral Density Matrix (frequency distribution of the energy)

- Our goal: Linearization with Random Parameters:

$$a = \text{(vector) random variable}$$

$$[\boldsymbol{M}(a)]^{eq}\ddot{\boldsymbol{Q}} + [\boldsymbol{C}(a)]^{eq}\dot{\boldsymbol{Q}} + [\boldsymbol{K}(a)]^{eq}\boldsymbol{Q} + \boldsymbol{B}^{eq}(a) = \boldsymbol{W}$$

1-D.O.F. CONSERVATIVE SYSTEMS

$$\ddot{q} + f(q) = 0,$$

$$f(q)\,(\text{regular enough}), \;\; q f(q) > 0 \;\; q \neq 0, \;\; f(q) = -f(-q)$$

-Amplitude-phase change:

$$\text{given } (a, \varphi), \;\; a > 0, \; \varphi \in [0, 2\pi], \text{ set}$$

$$q(t) = a \cos \phi(t)$$

$$\dot{\phi}(t) = \Omega(a, \phi(t)), \;\; \phi(0) = \varphi$$

- Find

$$\Omega(a, \phi) = \Omega(a, \phi + 2\pi).$$

- Substitution yields

$$\frac{d}{d\phi}\left(\Omega^2 \sin^2\phi\right) - \frac{2}{a}\sin\phi f(a\cos\phi) = 0$$

$$\Omega^2(a,\phi) = \frac{2\int_0^{\phi} \sin\sigma f(a\cos\sigma)\, d\sigma}{a\sin^2\phi} = \frac{2(V(a) - V(a\cos\phi))}{a^2\sin^2\phi} \qquad (>0)$$

$$T(a) = \int_0^{2\pi} \frac{1}{\Omega(a,\phi)}\, d\phi$$

- Properties

$$\Omega^2(a,\phi) = \Omega_0^2(a) + \sum_{i=1}^{\infty} \Omega_{2i}^2(a)\cos(2i\phi)$$

$$\Omega_0^2(a) = \frac{1}{a\pi}\int_0^{2\pi} f(a\cos\phi)\cos\phi\, d\phi = \frac{\mathcal{F}_1(a)}{a}$$

COMMENT

- Finite Fourier Series for polynomial: $f(q) = \omega^2 q + \lambda q^3, \quad \lambda > 0$

$$\Omega^2(a, \phi) = \omega^2 + \frac{3}{4}\lambda a^2 + \frac{1}{4}\lambda a^2 \cos(2\phi)$$

$$T(a) = \frac{4}{\sqrt{\omega^2 + \lambda a^2}}\mathcal{K}(k), \; k^2 = \frac{\lambda a^2}{2(\omega^2 + \lambda a^2)}$$

- Otherwise truncated Fourier series

RANDOM INPUT

$$\ddot{q}(t) + 2\varepsilon c\,\dot{q}(t) + f(q(t)) = 2\sqrt{\varepsilon}\;s\,\dot{w}(t) \quad (\dot{w}(t) = \text{unit white noise})$$

- van der Pol transform: with $\Omega(a, \phi)$ previously defined

$$(q, \dot{q}) \Leftrightarrow (a, \varphi)$$

$$q(t) \;=\; a(t)\cos\phi(t)$$

$$\dot{q}(t) = -\,a(t)\Omega(a(t), \phi(t))\sin\phi(t)$$

$$\dot{\phi}(t) = \Omega(a(t), \phi(t)) + \dot{\varphi}(t)$$

STRATONOVICH EQUATIONS

- Compatibility eq.

$$\dot{a}\cos\phi = a\,\dot{\varphi}\sin\phi$$

- Evolution eqs.

$$\dot{a} = -\,2\varepsilon\,c\,a^2\,\frac{\Omega^2(a,\phi)}{f(a)}\,\sin^2(\phi) - 2\sqrt{\varepsilon}\,s\,a\,\frac{\Omega(a,\phi)}{f(a)}\,\sin(\phi)\circ\,\dot{w}$$

$$\dot{\varphi} = \varepsilon\,c\,a\,\frac{\Omega^2(a,\phi)}{f(a)}\,\sin(2\phi) - 2\sqrt{\varepsilon}\,s\,\frac{\Omega(a,\phi)}{f(a)}\cos(\phi)\circ\,\dot{w}$$

$$\dot{\phi} = \Omega(a,\phi) + \dot{\varphi}$$

AVERAGING

Averaging = Removing oscillating terms: $< g(a) > = \frac{1}{2\pi} \int_0^{2\pi} g(a, \phi)\, d\phi$

(R.L.Stratonovich 1963 and 1967 *Topics in the theory of random noise*, Gordon and Breach)

$$\sigma(x(t)) \circ dw(t) = \sigma_x(x(t))\sigma(x(t))dt + \sigma(x(t))dw(t)$$

$$\mathbb{E}\,\sigma(x(t))dw(t) = 0, \ \mathbb{E}\,(\sigma(x(t))dw(t))^2 = \mathbb{E}\,(\sigma(x(t))^2\, dt$$

- After averaging one finds Itô's SDE equations of the form

$$da = \varepsilon m(a)\, dt + 2\sqrt{\varepsilon}\, n(a)\, dw_1 \ \ \text{(slowly varying)}$$

$$d\varphi = 2\sqrt{\varepsilon}\, h(a)\, dw_2\,, \quad (d\phi = < \Omega > dt + d\varphi)$$

Amplitude eq. do not depend on the phase eq. and $w_1 \perp w_2$

Fokker-Planck equation

- Stationary Fokker-planck eq.

$$\mathcal{L}^*(p) = \varepsilon \frac{\partial}{\partial a}\Big[- m(a)\,p + \frac{\partial}{\partial a}\,(n^2(a)\,p)\Big] = 0$$

$$- m(a)\,p + \frac{d}{da}\,(n^2(a)\,p) = 0 \;\rightarrow p(a) = \mathbb{C}\,\frac{1}{n^2(a)}\exp\int \frac{m(a)}{n^2(a)}\,da$$

- Linear case $f(q) = \omega^2 q, \quad p(a) = \frac{a\,c\,\omega^2}{s^2}\exp\big(-\frac{a^2 c\,\omega^2}{2\,s^2}\big),$ (Rayleigh)

- Phase φ (or ϕ) is uniformely distributed on $[0, 2\pi]$

- The r.v. $a(t), \varphi(t)$ are independent

$$p(a,\varphi) = \frac{1}{2\pi}\,p(a) \quad (= p(a,\phi))$$

EQUIVALENT LINEAR SYSTEM WITH RANDOM COEFFICIENTS

$$\ddot{q}(t) + 2\varepsilon c\,\dot{q}(t) + f(a(t)\cos\phi(t)) = 2\sqrt{\varepsilon}\; s\,\dot{w}(t)$$

$$f(a\cos\phi) = \mathcal{F}_1(a)\cos\phi + \sum_{n=2}^{\infty} \mathcal{F}_{2n+1}(a)\cos(2n+1)\phi = \frac{\mathcal{F}_1(a)}{a}\, q + \varepsilon r(\phi)$$

$$< \Omega^2(a,.) > = \Omega_0^2(a) = \frac{\mathcal{F}_1(a)}{a}$$

- ELS: Consider the slowly varying process $a(t)$ as a random variable a with density $p(a)$ (previously defined)

$$(\,*\,)\;\; \ddot{q}(t) + 2\varepsilon c\,\dot{q}(t) + \Omega_0^2(a)\,q(t) = 2\sqrt{\varepsilon}\; b(a)\, s\,\dot{w}(t)$$

$$(*) \; \ddot{q}(t) + 2\varepsilon c\, \dot{q}(t) + \Omega_0^2(a)\, q(t) = 2\sqrt{\varepsilon}\; b(a)\, s\, \dot{w}(t)$$

- Weighting function $b(a)$: preserving energy for given a

$$\mathbb{E}(q^2 \,|a) = \frac{1}{2\pi} \int_0^{2\pi} a^2 \cos^2(\phi)\, d\phi = \frac{a^2}{2}$$

- From $(*)$ follows

$$\mathbb{E}(q^2 \,|a) = \frac{b^2(a)\, s^2}{c\, \Omega_0^2(a)}, \text{ equating with } \frac{a^2}{2} \Rightarrow b^2(a) = \frac{a^2 c\, \Omega_0^2(a)}{2\, s^2}$$

RELATION WITH LINEARIZATION WITH CONSTANT COEFFICIENTS

-Recall

$$f(q) \sim \omega^2 q \;\; s.t. \;\; \min_{\omega^2} \; \mathbb{E}(f(q) - \; \omega^2 \, q \;)^2) \Rightarrow \mathbb{E}(f(q)q - \; \omega^2 \, q^2) = 0$$

- Let $q = a\cos\phi$ and $\mathbb{E}(.)$ calculated with $p(a,\phi) = \frac{1}{2\pi}\,p(a)$

$$\int_a \left(2 \int_\phi f(a\cos\phi)\cos\phi \, d\phi - \omega^2 \, a \right) \frac{a}{2} \, p(a) \, da = 0$$

$$\left(2 \int_\phi f(a\cos\phi)\cos\phi \, d\phi - \omega^2 \, a \right) = 0$$

$$\omega^2 = \frac{\mathcal{F}_1(a)}{a} = \Omega_0^2(a)$$

POWER SPECTRAL DENSITY (PSD)

$$\ddot{q}(t) + 2\varepsilon c\,\dot{q}(t) + \Omega_0^2(a)\,q(t) = 2\sqrt{\varepsilon}\; b(a)\, s\,\dot{w}(t)$$

- ELS: Random Transfert Function

$$H(\omega|a) = (2\varepsilon c\, i\omega - \omega^2 + \Omega_0^2(a))^{-1}$$

- ELS: conditional PSD

$$S_q(\omega|a) = |H(\omega|a)|^2\, 4\varepsilon\, b^2(a)\, s^2$$

- NLS: Approximate PSD

$$S_q(\omega) = \int_0^\infty S_q(\omega|a)\, p(a)\, da$$

DUFFING OSCILLATOR

$$\ddot{q} + 2\varepsilon\,\dot{q} + q + \lambda q^3 = 2\sqrt{\varepsilon}\;\dot{w}(t)$$

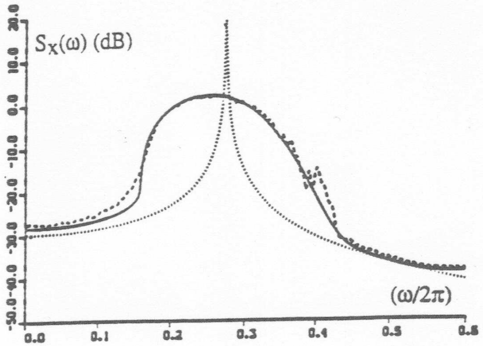

Figure 5. $f(q) = \lambda q^3, \varepsilon = 2.5\,10^{-3}, \;\; \lambda = 2.$

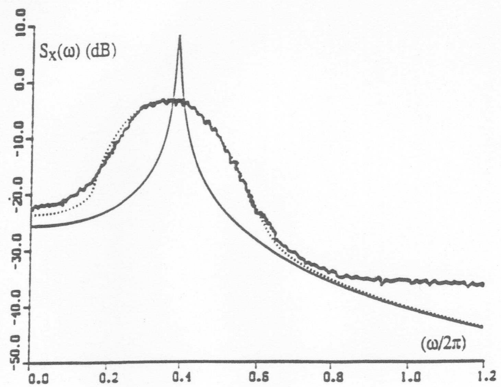

Figure 6. $\varepsilon = 2.5\,10^{-2}, \;\; \lambda = 10$

ELASTIC-STOPS-SYSTEM

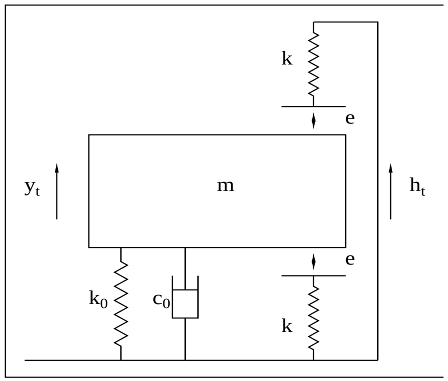

Figure 7.

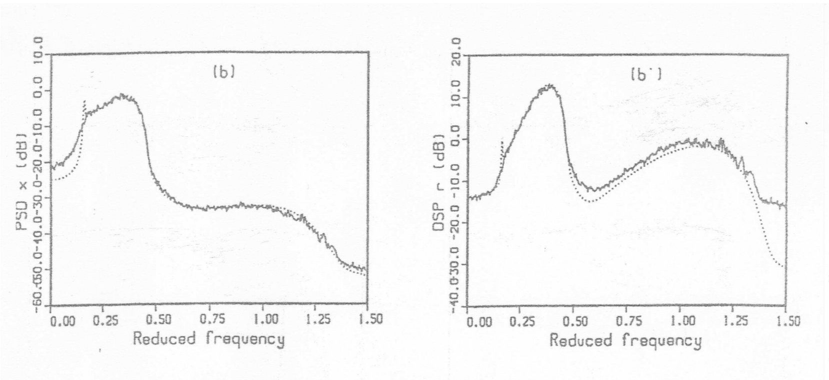

Figure 8. PSD: q (left), $f(q)$ (right)

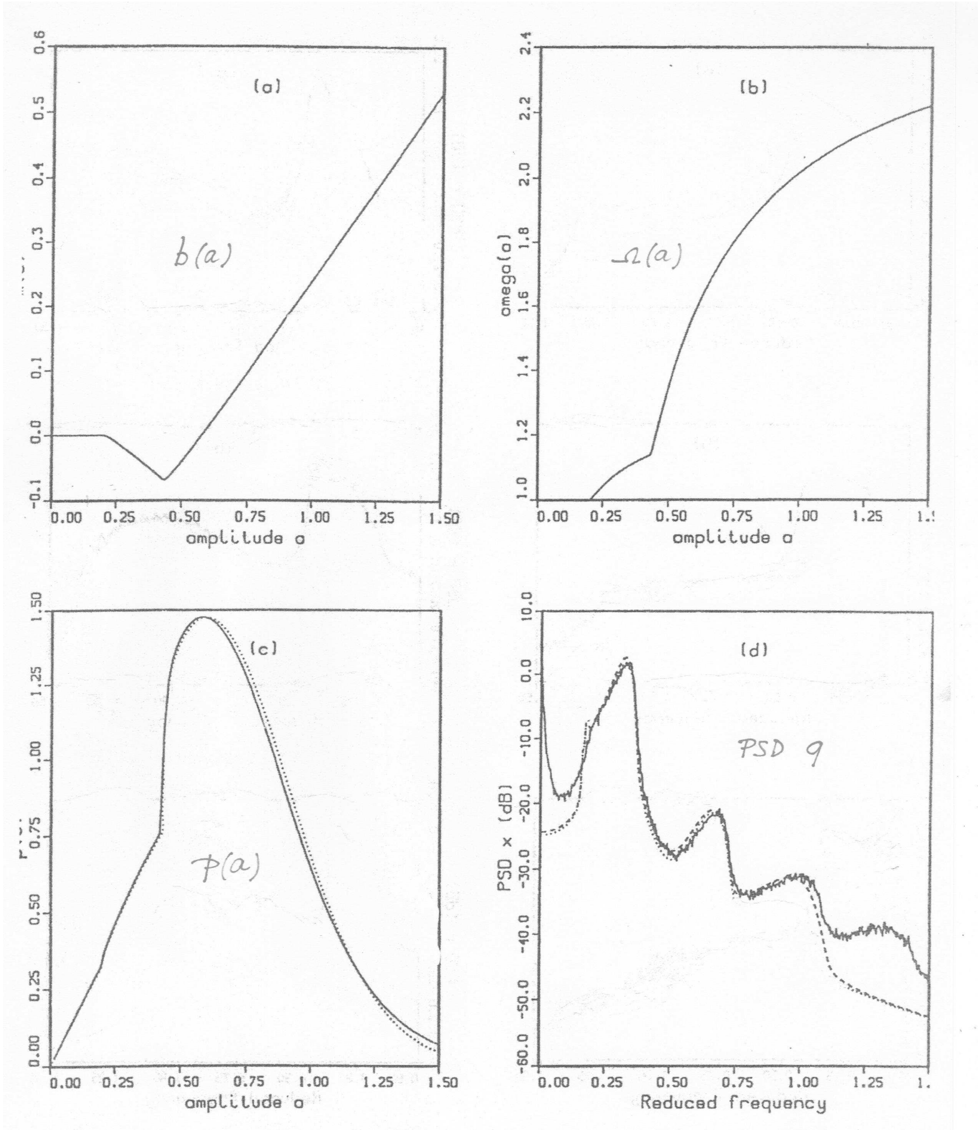

Figure 9. Asymmetrical clearance and stiffnesses

SUMMING UP

$$\ddot{q}(t) + 2\varepsilon c\,\dot{q}(t) + f(q(t)) = 2\sqrt{\varepsilon}\; s\,\dot{w}(t)$$

- Construct the nonlinear mode of the conservative system
- Perform a van der Pol transformation for the forced eq. based on the NL conservative mode
- Apply stochastic averaging to Stratonovich SDE for $(a(t), \varphi(t))$ and derive the p.d.f. $p(a)$
- Define an ELS with random parameter ($a(t) \sim a$ a r.v. with density $p(a)$)

$$\ddot{q}(t) + 2\varepsilon c\,\dot{q}(t) + \Omega_0^2(a)\,q(t) = 2\sqrt{\varepsilon}\; b(a)\, s\,\dot{w}(t)$$

- Define an approximate PSD for the non-linear response

$$S_q(\omega) = \int_0^{\infty} S_q(\omega|a)\, p(a)\, da$$

NON-LINEAR DAMPING AND ASYMMETRICAL CASE

$$\ddot{q}(t) + c(q(t), \dot{q}(t))\dot{q}(t) + f(q(t)) = 0$$

- Amplitude-phase change

$$\begin{aligned} q(t) &= v(t)(\cos\boldsymbol{\phi}(t) + b) \\ \dot{v}(t) &= v(t)\xi \qquad\qquad v(0) = a \\ \dot{\phi}(t) &= \Omega \qquad\qquad \phi(0) = \varphi \end{aligned}$$

- Dynamics

$$\xi(v, \phi) = \xi(v, -\phi) = \xi(v, \phi + \pi), \quad \Omega(v, \phi) = \Omega(v, -\phi) = \Omega(v, \phi + \pi)$$

- To balance quadratic terms

$$b(v, \phi) = b(v, \phi + \pi).$$

EXAMPLE

$$\ddot{q} + c(q)\,\dot{q} + q = 0 \quad \text{where} \quad c(q) = -\,\varepsilon(1 - \frac{22}{3}q^2 + 8q^4 - \frac{32}{15}q^6)$$

$$\frac{\partial \xi}{\partial \phi}\cos\phi - 2\xi\sin\phi - \xi v\,\frac{1}{2\Omega^2}\frac{\partial \Omega^2}{\partial v}\sin\phi = \varepsilon\,c(v\cos\phi)\sin\phi$$

$$\frac{1}{2}\frac{\partial \Omega^2}{\partial \phi}\sin\phi + \Omega^2\cos\phi - \cos\phi = \cos\phi\left(\xi^2 + \frac{1}{2}v\frac{\partial \xi^2}{\partial v} + \varepsilon\,c(v\cos\phi)\,\xi\right)$$

$$\left\{\begin{array}{lcl}\dot{\phi} & = & \Omega(v,\phi)\\ \dot{v} & = & v\xi(v,\phi)\end{array}\right. \Rightarrow \frac{d\,v}{d\phi} = v\,\frac{\xi(v,\phi)}{\Omega(v,\phi)}$$

- π-periodic solutions : $v^*(\phi) = v^*(\phi+\pi)$
- Limit cycles $q(t) = v^*(\phi(t))\,(\cos\,\phi(t))$ $\quad T^* = \int_0^{2\pi} \frac{d\phi}{(\Omega(v^*(\phi)),\,\phi)}$

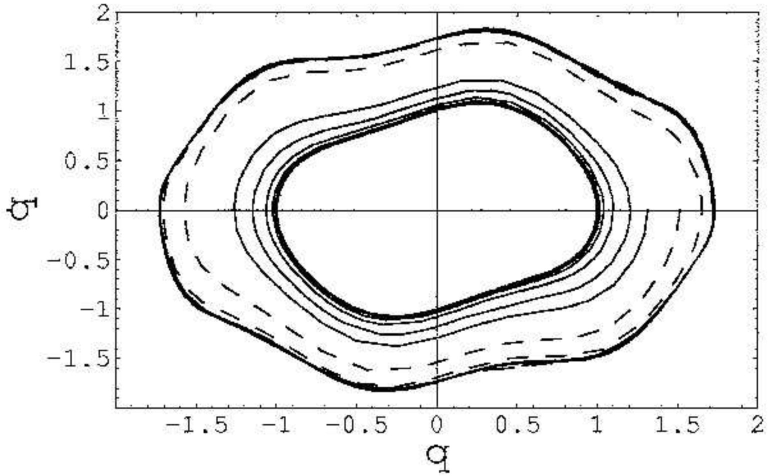

Figure 10. 3 limit cycles

	ε=0.01	ε=0.1	ε=0.5	ε=0.8	ε=1
T approx.	6.2832	6.2860	6.3543	6.4710	6.5833
T given by a shooting method	6.2832	6.2849	6.3263	6.3938	6.4564

Tableau 1. Period of the limit cycle v_1^*=1 for various values of ε.

MULTI-DEGREE OF FREEDOM: NONLINEAR NORMAL MODES

$$[\mathbf{M}]\ddot{\mathbf{Q}}(t) + \boldsymbol{G}(\dot{\boldsymbol{Q}}(t), \boldsymbol{Q}(t)) = \mathbf{0}, \;\; [\boldsymbol{M}] > 0$$

$$\boldsymbol{Q} = v\,(\boldsymbol{\Psi}_{\mathrm{c}}(v,\phi)\cos\phi - \boldsymbol{\Psi}_{\mathrm{s}}(v,\phi)\sin\phi) + v\,\boldsymbol{B}(v,\phi)$$

$$\begin{cases} \dot{\phi}(t) & = & \Omega(v(t), \phi(t))\;, \\ \dot{v}(t) & = & v(t)\xi(v(t), \phi(t))\;, \end{cases} \quad \phi(0) = \varphi, \;\; v(0) = a$$

$$\boldsymbol{\Psi}_{\mathrm{c}}^{T}[\mathbf{M}]\boldsymbol{\Psi}_{\mathrm{c}} + \boldsymbol{\Psi}_{\mathrm{s}}^{T}[\mathbf{M}]\boldsymbol{\Psi}_{\mathrm{s}} = 1, \;\; \boldsymbol{\Psi}_{\mathrm{c}}^{T}[\mathbf{M}]\boldsymbol{\Psi}_{\mathrm{s}} = 0$$

$$\Omega(v,\phi) = \Omega(v, -\phi) = \Omega(v, \phi+\pi), \;\; \xi(v,\phi) = \xi(v, -\phi) = \xi(v, \phi+\pi)$$

$$\boldsymbol{\Psi}_{.}(v,\phi) = \boldsymbol{\Psi}_{.}(v, -\phi) = \boldsymbol{\Psi}_{.}(v, \phi+\pi)$$

$$\boldsymbol{B}(v,\phi) = \boldsymbol{B}(v, \phi+\pi)$$

MULTI-DEGREE OF FREEDOM: COUPLED MODES

$$[\boldsymbol{M}]\ddot{\boldsymbol{Q}}(t) + 2\varepsilon[\boldsymbol{C}]\dot{\boldsymbol{Q}}(t) + \boldsymbol{F}(\boldsymbol{Q}(t)) = 2\sqrt{\varepsilon}[\boldsymbol{S}]\dot{\boldsymbol{W}}(t)$$

- $\forall \boldsymbol{X} \in \mathbb{R}^n$, $\forall \boldsymbol{Y} \in \mathbb{R}^n$ $\boldsymbol{Y} \neq 0$, $\boldsymbol{Y}^T([\partial_{\boldsymbol{X}}\boldsymbol{F}(\boldsymbol{X})] + [\partial_{\boldsymbol{X}}\boldsymbol{F}(\boldsymbol{X})]^T)\boldsymbol{Y} > 0$ (regular enough)
- $[\boldsymbol{M}]$ and $[\boldsymbol{C}]$ are positive definite symmetrical matrices. $[\boldsymbol{S}]$ an nxm real matrix.
- The eigenvalue problem $[\partial\boldsymbol{F}(\boldsymbol{X^o})]\boldsymbol{\Psi} = [\boldsymbol{M}]\boldsymbol{\Psi}\omega^2$ has n distinct eigenvalues (necessarily > 0) where $\boldsymbol{X^o}$ denotes the unique root of $\boldsymbol{F}(\boldsymbol{X}) = 0$.
- $\dot{\boldsymbol{W}}(t)$: $\mathbb{R}^m$-unit white noise ($\mathbb{E}(\dot{\boldsymbol{W}}(t)\dot{\boldsymbol{W}}(t+\tau)) = [\boldsymbol{I}]\delta(\tau)$).

- The NL system has a unique asymptotically stable second order stationary solution.

A VAN DER POL TRANSFORM

$$\boldsymbol{a} = (a_1\,, a_2\,, \ldots., a_n\,) \in \mathbb{R}^n_+, \quad \boldsymbol{\varphi} = (\varphi_1\,, \varphi_2\,, \ldots., \varphi_n\,) \in [0, 2\pi]^n$$

$$\boldsymbol{Q}(t) = \boldsymbol{B}(\boldsymbol{a}(t)) + \sum_{i=1}^{n} \; a_i(t)\boldsymbol{\Psi}_i(\boldsymbol{a}(t)) \cos \phi_i(t)$$

$$\dot{\boldsymbol{Q}}(t) = -\sum_{i=1}^{n} \; a_i(t)\, \Omega_i(\boldsymbol{a}(t))\; \boldsymbol{\Psi}_i(\boldsymbol{a}(t)) \sin \phi_i(t)$$

$$\dot{\phi}_i(t) = \Omega_i(\boldsymbol{a}(t)) + \dot{\varphi}_i(t), \;\; i = 1, 2, \ldots, n$$

$$\boldsymbol{\Psi}_i^T(\boldsymbol{a}(t))[\boldsymbol{M}]\boldsymbol{\Psi}_i(\boldsymbol{a}(t)) = 1$$

Only the coupling through the amplitudes is taken in account

DEFINITION OF $\boldsymbol{B}(\boldsymbol{a}), \boldsymbol{\Psi}_i(\boldsymbol{a}),\ \Omega_i^2(\boldsymbol{a}),\ i=1,2,\ldots,n$

- With $\boldsymbol{Q}=\boldsymbol{B}(\boldsymbol{a})+\sum_{i=1}^{n} a_i \boldsymbol{\Psi}_i(\boldsymbol{a}) \cos\phi_i$ consider the Fourier series

$$\boldsymbol{F}(\boldsymbol{Q})=\mathcal{F}_0(\boldsymbol{a},\boldsymbol{B},\boldsymbol{\Psi})+\sum_{i=1}^{n} \mathcal{F}_i(\boldsymbol{a},\boldsymbol{B},\boldsymbol{\Psi}) \cos\phi_i+\varepsilon \boldsymbol{r}(\phi_1,\phi_2\ldots\phi_n)$$

- Modal functions: for each $\boldsymbol{a} \in \mathbb{R}_+^n$, find $(\boldsymbol{B}(\boldsymbol{a}),\ \boldsymbol{\Psi}_i(\boldsymbol{a}),\ \Omega_i^2(\boldsymbol{a}),\ i=1,n)$ such that

$$\mathcal{F}_0(\boldsymbol{a},\boldsymbol{B},\boldsymbol{\Psi})=0$$

$$[\mathrm{M}]\boldsymbol{\Psi}_i\,\Omega_i^2=\frac{1}{a_i}\mathcal{F}_i(\boldsymbol{a},\boldsymbol{B},\boldsymbol{\Psi}) \quad i=1,2,\ldots,n$$

$$\boldsymbol{\Psi}_i^T(\boldsymbol{a})[\mathrm{M}]\boldsymbol{\Psi}_i(\boldsymbol{a})=1$$

- Integration by parts

$$[\mathbf{M}]\boldsymbol{\Psi}_i\,\Omega_i^2 = \frac{2}{(2\pi)^n}\int_0^{2\pi}\dots\int_0^{2\pi}\,[\partial_{\boldsymbol{Q}}\boldsymbol{F}(\boldsymbol{Q})]\sin^2\phi_i\;d\boldsymbol{\phi}\,\boldsymbol{\Psi}_i(\boldsymbol{a}) \quad (\Rightarrow \Omega_i^2(\boldsymbol{a})>0)$$

- The vectors $\boldsymbol{\Psi}_j(\boldsymbol{a})$ do not verify in general the orthogonality properties.

Adjoint modal vectors:

$$\boldsymbol{\Gamma}_i^T(\boldsymbol{a})[\mathbf{M}]\boldsymbol{\Psi}_j(\boldsymbol{a}) = \delta_{ij}$$

- Unique solution in a neighborhood of $\boldsymbol{a}=0$ (deformation of modes issued from $[\mathbf{M}]\boldsymbol{\Psi}_i\,\Omega_i^2 = [\partial_{\boldsymbol{X}}\boldsymbol{F}(\boldsymbol{X}_0)]\boldsymbol{\Psi}_i$)).

AN ANALYTICAL SOLUTION

- Radial potential: $\boldsymbol{F}(\boldsymbol{Q}) = \partial_{\boldsymbol{Q}} V(\frac{1}{2}\boldsymbol{Q}^T[\boldsymbol{K}]\boldsymbol{Q})$, $[\boldsymbol{K}]$ sym > 0,

$$\boldsymbol{B} = 0$$

$$[\mathbf{M}]\boldsymbol{\Psi}_i\,\omega_i^2 = [\boldsymbol{K}]\boldsymbol{\Psi}_i\,, \quad \boldsymbol{\Psi}_i{}^T[\mathbf{M}]\boldsymbol{\Psi}_i = 1, \;\; \boldsymbol{\Gamma}_i = \boldsymbol{\Psi}_i \text{ do not depend on } \boldsymbol{a}$$

$$\Omega_i^2(\boldsymbol{a}) = \frac{2\omega_i^2}{(2\pi)^n}\int_0^{2\pi}\dots\int_0^{2\pi} \partial V(\frac{1}{2}\,\boldsymbol{Q}^T[\boldsymbol{K}]\boldsymbol{Q})\cos^2\phi_i\; d\phi_1 \dots d\phi_n$$

- If $V(x) = |x| + x^2$, and , $\Omega_i^2(\boldsymbol{a}) = \omega_i^2\,(1 + \sum_j \alpha_{ij}^2\,\omega_j^2\; a_j^2)$

NUMERICAL EXAMPLE

$$[\boldsymbol{M}]\ddot{\boldsymbol{Q}}(t) + 2\varepsilon[\boldsymbol{C}]\dot{\boldsymbol{Q}}(t) + [\boldsymbol{K}_0]\boldsymbol{Q}(t) + (\boldsymbol{Q}^T(t)[\boldsymbol{K}_1]\boldsymbol{Q}(t))\boldsymbol{Q}(t) = 2\sqrt{\varepsilon}[\boldsymbol{S}]\dot{\boldsymbol{W}}(t)$$

$$\varepsilon = 0.02, \qquad [\boldsymbol{M}] = \begin{pmatrix} 1 & 0.25 \\ 0.25 & 1 \end{pmatrix}, \; [\boldsymbol{S}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[\boldsymbol{C}] = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \; [\boldsymbol{K}_0] = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \; [\boldsymbol{K}_1] = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

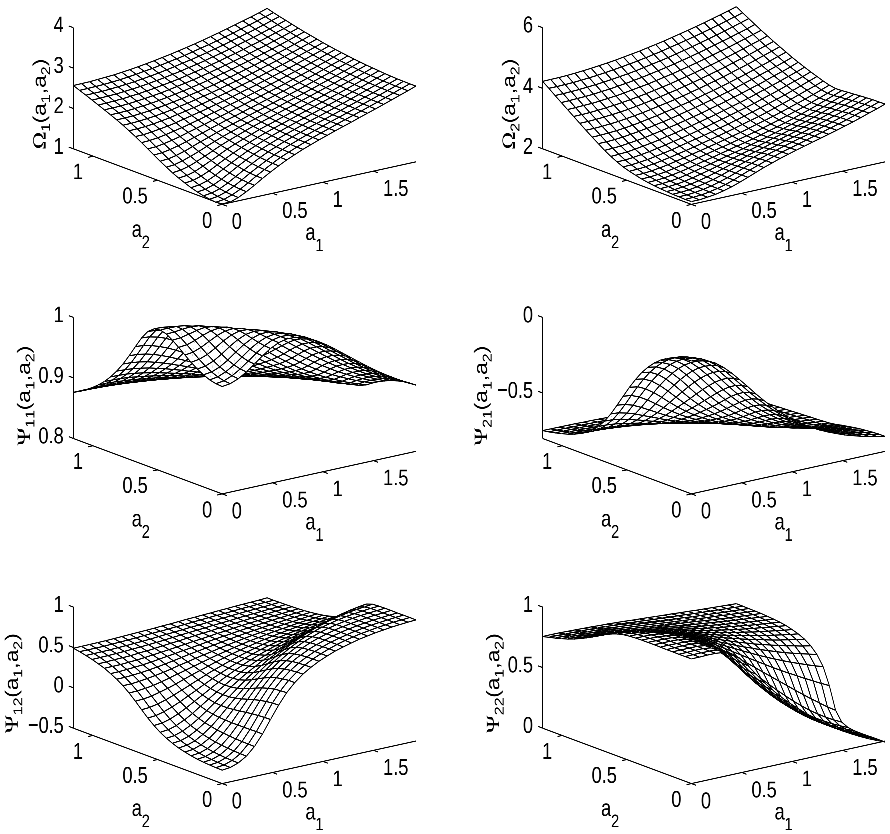

Figure 11.

SDE, AVERAGING, FOKKER-PLANCK

$$([\mathbf{1}] + \varepsilon[\boldsymbol{\alpha}])\dot{\boldsymbol{a}} = [\mathrm{diag}\,\frac{\sin\boldsymbol{\phi}}{\boldsymbol{\Omega}}](2\varepsilon[\boldsymbol{\Gamma}^T][\boldsymbol{C}]\dot{\boldsymbol{Q}} + \varepsilon\boldsymbol{r}) - 2\sqrt{\varepsilon}\,[\mathrm{diag}\,\frac{\sin\boldsymbol{\phi}}{\boldsymbol{\Omega}}][\boldsymbol{\Gamma}^T][\boldsymbol{S}]\circ\dot{\boldsymbol{W}}$$

$$\varepsilon[\boldsymbol{\beta}]\dot{\boldsymbol{a}} + [\mathbf{1}]\dot{\boldsymbol{\varphi}} = [\mathrm{diag}\,\frac{\cos\boldsymbol{\phi}}{\boldsymbol{\Omega}}](2\varepsilon[\boldsymbol{\Gamma}^T][\boldsymbol{C}]\dot{\boldsymbol{Q}} + \varepsilon\boldsymbol{r}) - 2\sqrt{\varepsilon}\,[\mathrm{diag}\,\frac{\cos\boldsymbol{\phi}}{\boldsymbol{a}\boldsymbol{\Omega}}][\boldsymbol{\Gamma}^T][\boldsymbol{S}]\circ\dot{\boldsymbol{W}}$$

$$\int_{\boldsymbol{\phi}}[\boldsymbol{\alpha}(\boldsymbol{a},\boldsymbol{\phi})]d\phi_1\ldots d\phi_n = [\bar{\boldsymbol{\alpha}}(\boldsymbol{a})], \qquad \int_{\boldsymbol{\phi}}[\boldsymbol{\beta}(\boldsymbol{a},\boldsymbol{\phi})]d\phi_1\ldots d\phi_n = [\mathbf{0}]$$

After Averaging no exact solution of the associated Fokker-Planck equation

AN APPROXIMATE SOLUTION

- Matrices $[\boldsymbol{\alpha}(\boldsymbol{a}, \boldsymbol{\phi})]$ and $[\boldsymbol{\beta}\,(\boldsymbol{a}, \boldsymbol{\phi})]$ are replaced by their mean value

$$da_i = \varepsilon\, m_i(\boldsymbol{a})\, dt + 2\sqrt{\varepsilon}\, n_i(\boldsymbol{a})\, dw_i \quad \text{with} \quad w_i \perp w_j$$

- Stationary Fokker-Planck equation reads $\mathcal{L}^*(p) = \sum_{i=1}^{n} \mathcal{L}_i^*(p) = 0$

$$\mathcal{L}_i^*(p) = \varepsilon \frac{\partial}{\partial a_i} \left[-m_i(\boldsymbol{a})\, p + \frac{\partial}{\partial a_i} \left(n_i^2(\boldsymbol{a})\, p \right) \right]$$

- Define

$$q_i = \text{non-normalized solution to } \mathcal{L}_i^*(q_i) = 0 \quad \text{for fixed } a_{j \neq i}$$

$$p(a_1, a_2, \ldots, a_n) = \mathbb{C}_N\, q_1(a_1|a_2, \ldots, a_n)\, q_2(a_2|a_1, \ldots, a_n) \ldots . q_n(a_n|a_1, \ldots, a_{n-1})$$

$$\mathbb{C}_N = \text{constant of (re)-normalization}$$

Typically

$$q_i(\boldsymbol{a}) = a_i\, \Omega_i(\boldsymbol{a})\, f(\boldsymbol{a}, s_i(\boldsymbol{a}))\, \exp\big(- \int_0^{a_i} c_i(\boldsymbol{\alpha}) \Omega_i^2(\boldsymbol{\alpha})\, g(\boldsymbol{\alpha}, s_i(\boldsymbol{\alpha}))\, \alpha_i\, d\alpha_i\big)$$

$$c_i(\boldsymbol{a}) = \boldsymbol{\Gamma}_i^T(\boldsymbol{a})[\boldsymbol{C}]\boldsymbol{\Psi}_i(\boldsymbol{a}),\;\; s_i^2(\boldsymbol{a}) = \boldsymbol{\Gamma}_i^T(\boldsymbol{a})[\boldsymbol{S}][\boldsymbol{S}]^T\boldsymbol{\Gamma}_i(\boldsymbol{a})$$

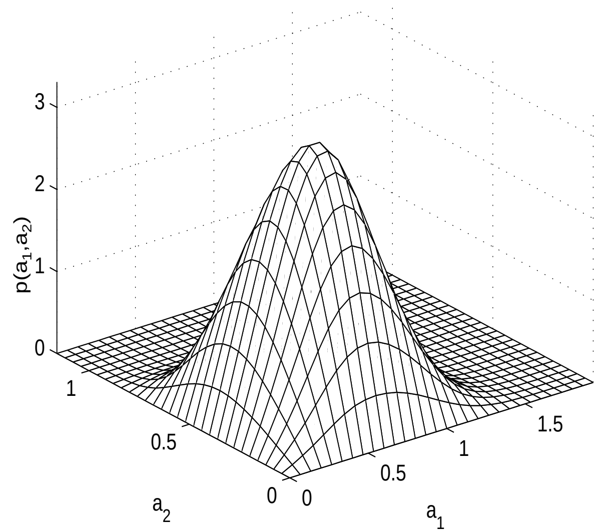

Figure 12.

MODAL ELS

- $\boldsymbol{a}$ is now a vector r.v. with p.d.f as above

$$\boldsymbol{Q}(t) = \boldsymbol{B}(\boldsymbol{a}) + \sum_{i=1}^{n} \boldsymbol{\Psi}_i(\boldsymbol{a})\, q_i(t)$$

- By applying $\boldsymbol{\Gamma}_i^T(\boldsymbol{a})$ to the NLS

$$\ddot{q}_i + 2\varepsilon \boldsymbol{\Gamma}_i^T [\boldsymbol{C}] \sum_{j=1}^{n} \boldsymbol{\Psi}_i\, \dot{q}_j + \Omega_i^2\, q_i + \boldsymbol{\Gamma}_i^T \varepsilon \boldsymbol{r} = 2\sqrt{\varepsilon}\, \boldsymbol{\Gamma}_i^T [\boldsymbol{S}] \boldsymbol{W}(t)$$

- Small damping, white noise input, $\Rightarrow$ modal ELS (as 1 DOF)

$$\ddot{q}_i + 2\varepsilon\, c_i(\boldsymbol{a})\, \dot{q}_j + \Omega_i^2(\boldsymbol{a})\, q_i = 2\sqrt{\varepsilon}\, b_i(\boldsymbol{a}) \boldsymbol{\Gamma}_i^T(a)[\boldsymbol{S}] \boldsymbol{W}(t), \quad i = 1, n$$

$$\frac{\boldsymbol{a}^2}{2} = \frac{b_i^2(\boldsymbol{a})\, s_i^2(\boldsymbol{a})}{c_i(\boldsymbol{a}) \Omega_i^2(\boldsymbol{a})}, \quad c_i = \boldsymbol{\Gamma}_i^T [\boldsymbol{C}] \boldsymbol{\Psi}_i, \quad s_i^2 = \boldsymbol{\Gamma}_i{}^T [\boldsymbol{S}][\boldsymbol{S}]^T [\boldsymbol{\Gamma}_i$$

POWER SPECTRAL DENSITY MATRIX

$$\boldsymbol{q} = (q_1\,, q_2, , ..., q_n),\;\; [\boldsymbol{S}_{\boldsymbol{q}}(\omega;\boldsymbol{a})] = [\boldsymbol{H}(\omega;\boldsymbol{a})][\boldsymbol{\Sigma}(\boldsymbol{a})][\boldsymbol{H}(\omega;\boldsymbol{a})]^*$$

$$[\boldsymbol{\Sigma}(\boldsymbol{a})] = [\mathrm{diag}\, b_i(\boldsymbol{a})][\boldsymbol{\Gamma}^T(\boldsymbol{a})][\boldsymbol{S}][\boldsymbol{S}]^T[\boldsymbol{\Gamma}(\boldsymbol{a})][\mathrm{diag}\, b_i(\boldsymbol{a})]$$

$$\tilde{\boldsymbol{Q}} = \boldsymbol{Q} - \boldsymbol{B} = [\boldsymbol{\Psi}(\boldsymbol{a})]\boldsymbol{q}$$

$$[\boldsymbol{S}_{\tilde{\boldsymbol{Q}}}(\omega;\boldsymbol{a})] = \boldsymbol{\Psi}(\boldsymbol{a})][\boldsymbol{S}_{\boldsymbol{q}}(\omega;\boldsymbol{a})]\boldsymbol{\Psi}(\boldsymbol{a})]^T$$

$$[\boldsymbol{S}_{\tilde{\boldsymbol{Q}}}(\omega)] = \int_{\mathbb{R}^n_+} [\boldsymbol{S}_{\tilde{\boldsymbol{Q}}}(\omega;\boldsymbol{a})]p(\boldsymbol{a})\,d\boldsymbol{a}$$

DOMINANT MODES

In general, for vibrating mechanical systems, the first few modes (strictly smaller than n) suffice to correctly describe the response.

- m-dominant modes $m < n$ (selected by conventional linéarization with constant coefficients)

$$\boldsymbol{Q}_m = \boldsymbol{B}(\boldsymbol{a}) + \sum_{i=1}^{m} \boldsymbol{\Psi}_i(\boldsymbol{a})\, a_i \cos\phi_i$$

$$\int_{\boldsymbol{\phi}} \boldsymbol{F}(\boldsymbol{Q}_m)\ d\phi_1 \ldots d\phi_n = 0$$

$$[\mathbf{M}]\,\boldsymbol{\Psi}_i\,\Omega_i^2 = \left(2 \int_{\boldsymbol{\phi}} [\partial_{\boldsymbol{Q}} \boldsymbol{F}(\boldsymbol{Q}_m)] \sin^2\phi_i\ d\boldsymbol{\phi} \right) \boldsymbol{\Psi}_i(\boldsymbol{a}),\ \ i \leqslant m$$

- Other modes (conventional eigenvalue-problem)

$$[\mathbf{M}]\boldsymbol{\Psi}_i\,\Omega_i^2 = \left(\int_{\boldsymbol{\phi}} \left[\partial_{\boldsymbol{Q}} \boldsymbol{F}(\boldsymbol{Q}_m)\right]\, d\boldsymbol{\phi} \right) \boldsymbol{\Psi}_i(\boldsymbol{a}),\;\; m < i \leqslant n$$

- With $\boldsymbol{\Gamma}_i{}^T(\boldsymbol{a})[\mathbf{M}]\boldsymbol{\Psi}_j(\boldsymbol{a}) = \delta_{ij}$ and $p(\boldsymbol{a}) = p(a_1, a_2, ...a_m)$

$$[\boldsymbol{S}_{\boldsymbol{Q}}(\omega)] = \int_{\mathbb{R}_+^m} [\boldsymbol{S}_{\boldsymbol{Q}}(\omega;\boldsymbol{a})]p(\boldsymbol{a})\,da_1...da_m$$

NON-LINEAR PLATES (FROM REINHALL, MILES,...)

The accuracy of fatigue analysis of large complex structures can be greatly improved by considering the geometrical nonlinearities of plates and shells when subjected to high acoustic or vibratory loading. Life predictions based on the linear plate theory have been shown to deviate significantly from experiment in cases of intensive excitation.

Non-linear random vibration of a periodically stiffened plate is considered by Reinhall and Miles [JSV 1989 JVA 1997], Mei and Wentz [1982 experimental]. Starting with von Karman plate theory and the use of Pujara's space harmonics [JSV 1971] the authors obtain the nonlinear deflection u as an n-modes expansion:

$$u(x, y, t) = \sum_{i=1}^{n} q_i(t) \boldsymbol{\psi}_i(x, y)$$

in which $\boldsymbol{\psi}_i(x, y)$ is the ith "linear" eigenfunction and $q_i(t),\ \ i = 1, n$:

$$\ddot{q}_i + 2\varepsilon\omega_i\dot{q}_i + \omega_i^2 q_i + \sum_{j,k,l}^{n} c_i^{jkl} q_j q_k q_l = 2\sqrt{\varepsilon} S_i\, w(t)$$

where $0 < \varepsilon << 1$ and $w(t)$ is a wide-band stationary random excitation with zero mean (here a unit-white-noise).

RESULTS

A 3-mode expansion model ($n = 3$) is considered. The system parameter values can be found in Reinhall and Miles [JSV 1989].

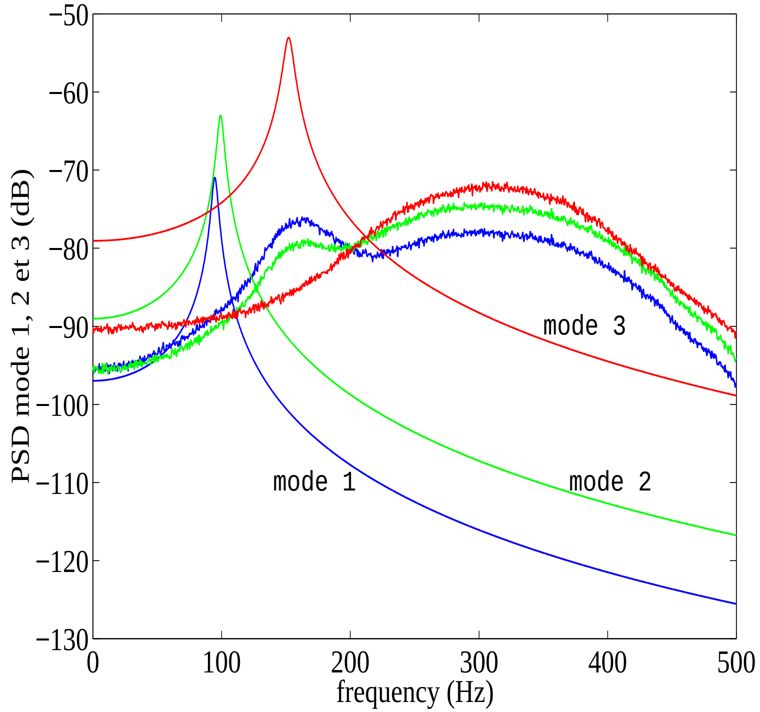

Figure 13. MC simulations: L and NL modes

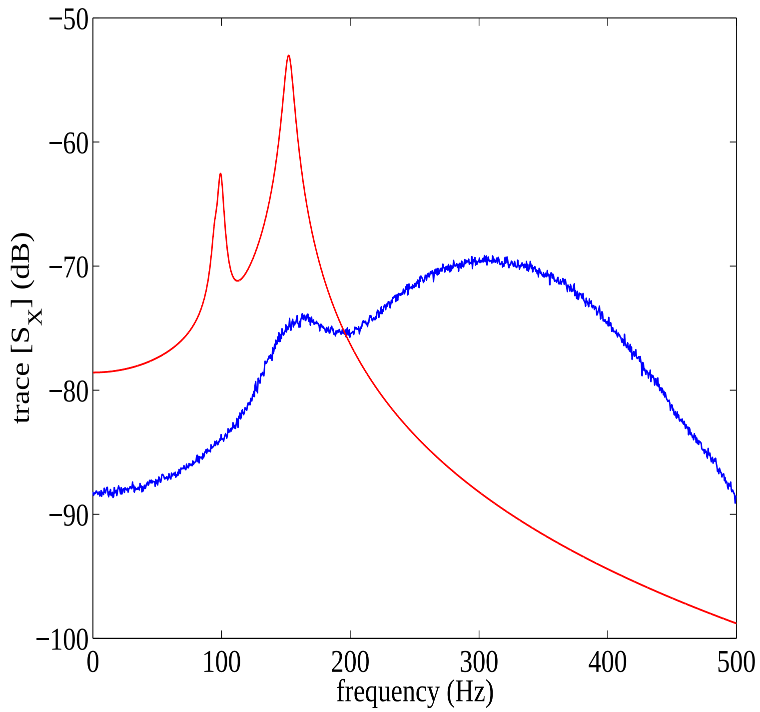

Figure 14. MC simulations

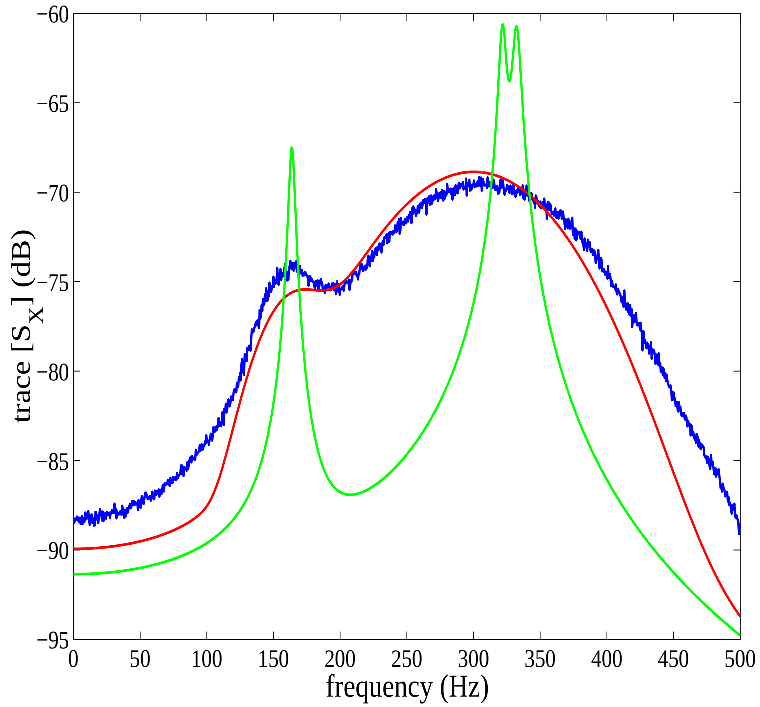

Figure 15. GSLM (green), Present M (red), MC (blue)

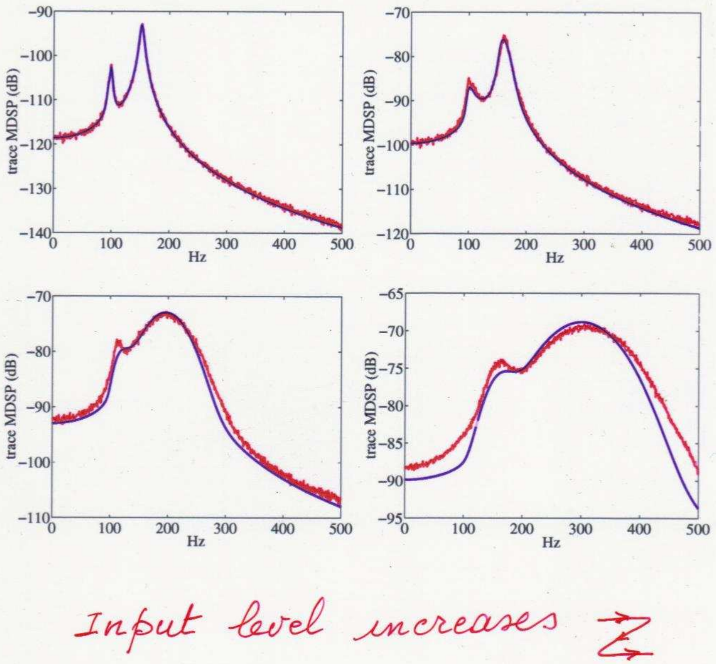

Figure 16.

DRIFTING RESPONSE OF HYSTERETIC OSCILLATOR

We discuss the drifting response of the elastic perfectly plastic hysteretic oscillator under zero mean random load with power spectrum vanishing at zero frequency.

When the excitation power spectrum is non zero at zero frequency, stochastic linearization predicts a displacement variance increasing linearly with time. When the excitation power spectrum vanishes at zero frequency, which is the case for several spectra used in earthquake engineering to model the ground motion, Monte Carlo simulations show (i) that the velocity power spectrum is non zero at zero frequency, and (ii) that in this case also the displacement variance linearly increases with time, whereas classical stochastic linearization predicts a bounded displacement variance, failing altogether to capture the physics of the response.

MODEL

$$\ddot{x}(t) + z(t) \;=\; 2\sqrt{\varepsilon}\,\sigma w(t)$$

$$\dot{z}(t) \;=\; \dot{x}(t)(1 - \mathbb{U}(z(t), \operatorname{sgn}\dot{x}(t)))$$

- $w(t) =$ zero mean wide-band stationary process with spectrum vanishing at zero frequency
- $\mathbb{U}(z, \operatorname{sgn}\dot{x}) = \mathbb{H}(|z| - 1)\mathbb{H}(z\dot{x})$, EPP model, $\mathbb{H}(u)$=Heaviside function
- $\mathbb{U}(z, \operatorname{sgn}\dot{x}) = \beta(\operatorname{sgn}\dot{x})|z|^{n-1}z + \gamma|z|^{n}$ smooth model $\beta > 0$
- Zero linear residual stiffness

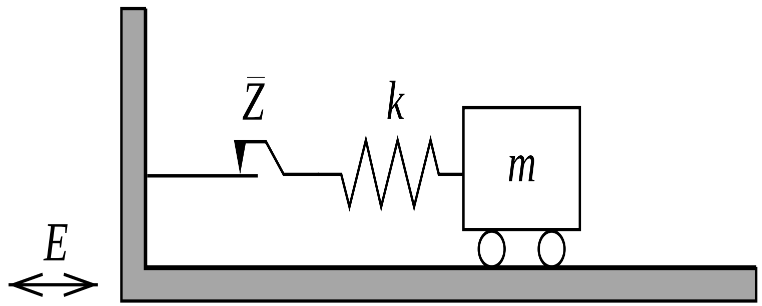

Figure 17. EPP oscillator under base excitation

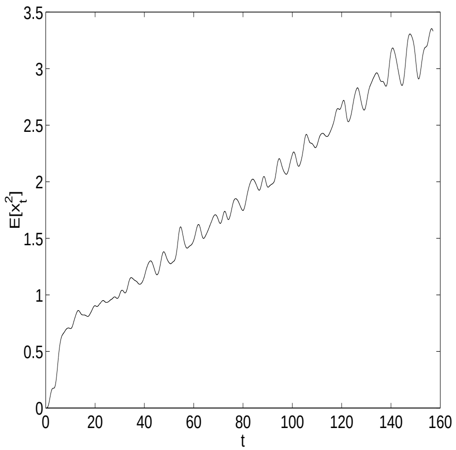

Figure 18. Mean-square displacement $2\sqrt{\varepsilon}\sigma = 0.5$ (MC), $S_w(0) = 0$

GAUSSIAN LINEARIZATION

$$\ddot{x}(t) + z(t) \;=\; 2\sqrt{\varepsilon}\,\sigma\, w(t)$$

$$\dot{z}(t) = \xi\, z(t) + \omega_0^2\, \dot{x}(t) \;\text{ where } \xi < 0$$

- $(z(t), \dot{x}(t))$ zero mean st. pr. $x(t) = \int_0^t \dot{x}(s)\, ds\,?$

$$S_{\dot{x}}(\omega) = 4\varepsilon\sigma^2 |H(\omega, \xi, \omega_0|^2\, S_w(\omega) \qquad (S_w(0) = 0 \Rightarrow S_{\dot{x}}(0) = 0)$$

$$\mathbb{E}(x^2(t)) = \frac{t}{\pi} \int_0^\infty S_{\dot{x}}(\nu/\pi t)\, \frac{\sin^2\nu}{\nu^2}\, d\nu \sim t\, S_{\dot{x}}(0) \text{ as } t \to \infty \;\; = 0$$

- Same result with a global linearization with random coefficients.

Our goal is to prove that $S_{\dot{x}}(0) \neq 0$

AMPLITUDE-PHASE TRANSFORM (1)

$$x = v\cos\phi + \int_0^t \hat{u}(s)\,ds$$

$$\dot{x} = y + \hat{u} \quad \text{where} \quad y = v(\xi(v)\cos\phi - \Omega(v))\sin\phi$$

$$\dot{v} = \xi(v)v + \dot{a}, \quad \dot{\phi} = \Omega(v) + \dot{\varphi}, \qquad (\xi(v) < 0 \text{ and } \Omega(v) > 0)$$

$$z = \tilde{z}(v,\phi) + \hat{z}$$

- v as a parameter: $\tilde{z}(v,\phi) = \tilde{z}(v,\phi + 2\pi)$ (cyclic component)

$$\Omega\frac{d\,\tilde{z}}{d\phi} = y\,(1 - \mathbb{U}(\tilde{z},\,\mathrm{sgn}\,y))$$

$$\tilde{z}(v,\phi) = h^c(v,\xi/\Omega)\cos\phi + h^s(v,\xi/\Omega)\sin\phi + \varepsilon\, r_{\tilde{z}}(\phi)$$

- $(a\ (\rightarrow\ v),\ \varphi\ (\rightarrow\ \phi),\ \hat{u},\ \hat{z})$ for 3 old variables $(x, \dot{x}, z)$

AMPLITUDE-PHASE TRANSFORM (2)

$$\dot{a}\,\cos\phi = \dot{\varphi}\,v\,\sin\phi \quad \text{(compatibility eq.)}$$

$$\sin\phi\,[(\dot{x})_v\,v\xi + (\dot{x})_\phi\,\Omega + h^c(v,\tau)\cos\phi + h^s(v,\tau)\sin\phi] +$$

$$+\sin\phi\,[\dot{\hat{u}} + \hat{z} + \varepsilon r_H + (v\varepsilon\xi_v\cos\phi - v\Omega_v\sin\phi)\dot{a}] = \Omega\,\dot{a} + 2\sqrt{\varepsilon}\,\sigma\sin\phi\,w$$

$$\dot{\hat{z}} + \dot{\tilde{z}} = (y+\hat{u})(1-\mathbb{U}(\tilde{z}+\hat{z}\,,\mathrm{sgn}(y+\hat{u})))$$

- Setting [....] $= 0 \Rightarrow$ expressions for $\xi(v), \Omega(v)$

$$\Omega^2 \sim h^c(v,0)/v \quad (>0)$$

$$2\xi\Omega \sim h^s(v,0)/v \; (\leqslant 0)$$

- Setting [....] $= 0 \;\Rightarrow$ closure equation

$$\dot{\hat{u}} + \hat{z} = -\,\varepsilon r_{\tilde{z}} - (v\xi_v \cos\phi - v\Omega_v \sin\phi)\dot{a}$$

- It remains

$$\dot{a} = -\,2\sqrt{\varepsilon}\,\frac{\sigma}{\Omega}\sin\phi\, w, \quad \dot{\varphi} = -\,2\sqrt{\varepsilon}\,\frac{\sigma}{v\Omega}\,\cos\phi\, w$$

STOCHASTIC AVERAGING

From the (a,φ)-eqs. follows (with $\xi = \varepsilon\tilde{\xi} < 0$)

$$\dot{v} = \varepsilon\tilde{\xi}\, v - 2\sqrt{\varepsilon}\,\frac{\sigma}{\Omega}\, w \sin\phi, \qquad \dot{\phi} = \Omega - 2\sqrt{\varepsilon}\frac{\sigma}{v\Omega}\, w\cos\phi$$

- After averaging ($\dot{w}$ a unit white noise)

$$\dot{v} = \varepsilon\tilde{\xi} v + \varepsilon\,\frac{\sigma^2}{\Omega}\,(S_w(\Omega)/\Omega)_v + \varepsilon\,\frac{\sigma^2}{v\Omega^2} S_w(\Omega) + \sqrt{2\varepsilon}\;\frac{\sigma}{\Omega}\,\sqrt{S_w(\Omega)}\;\dot{w}\,,$$

$$p_v(v) \;=\; \frac{v\Omega(v)\exp\Big[\frac{1}{\varepsilon\sigma^2}\int_0^v \varepsilon\tilde{\xi}(\alpha)\frac{\Omega^2(\alpha)}{S_e(\Omega(\alpha))}\,\alpha\, d\alpha\Big]}{\int_0^\infty v\Omega(v)\exp\Big[\frac{1}{\varepsilon\sigma^2}\int_0^v \varepsilon\tilde{\xi}(\alpha)\frac{\Omega^2(\alpha)}{S_e(\Omega(\alpha))}\,\alpha\, d\alpha\Big] dv}$$

$$p_{v\varphi} \;=\; \frac{1}{2\pi}\, p_v = p_{v\Phi}$$

SPECTRAL DENSITY OF $\hat{u}$

$$\dot{\hat{z}} + \dot{\tilde{z}} = (y + \hat{u})(1 - \mathbb{U}(\tilde{z} + \hat{z}, \operatorname{sgn}(y + \hat{u})))$$

$$\Updownarrow$$

$$\dot{\hat{z}} + \dot{\tilde{z}} = \varepsilon\tilde{\xi}(v)(\tilde{z} + \hat{z}) + \Omega^2(v)(y + \hat{u})$$

$$\dot{\hat{z}} - \varepsilon\tilde{\xi}\,\hat{z} - \Omega^2\,\hat{u} = 2\sqrt{\varepsilon}\,Y_2\,\sigma\,w + \varepsilon\tilde{r}_{\tilde{z}}$$

$$\dot{\hat{u}} + \hat{z} = 2\sqrt{\varepsilon}\,Y_1\,\sigma\,w + \varepsilon r_{\tilde{z}}$$

$$Y_2 = (\tilde{z}_v\,\sin\phi + \frac{\tilde{z}_\phi}{v}\cos\phi)/\Omega,\;\; Y_1 = \sin\phi(\varepsilon\xi_v\,v\cos\phi - v\,\Omega_v\sin\phi)/\Omega$$

$$v\,,\varphi \;\; \text{r.v. } \;\text{with } p(v,\varphi) = \frac{1}{2\pi}\, p(v) \;\; \text{and } \;\phi = \Omega(v)t + \varphi$$

$$S_u(0|v) = \frac{\varepsilon\sigma^2}{\Omega(v)^2} \sum_{n=-\infty}^{\infty} \mid Y_{2,2n}(v) - \varepsilon\tilde{\xi}(v) Y_{1,2n}(v) \mid^2 S_w(2n\Omega(v))$$

$$Y_2 = (\tilde{z}_v \,\sin\phi + \frac{\tilde{z}_\phi}{v} \cos\phi)/\Omega,\;\; Y_1 = \sin\phi\,(\varepsilon\xi_v\, v\cos\phi - v\Omega_v \sin\phi)/\Omega$$

$$S_u(0) = \int_0^\infty S_u(0|v)\, p(v)\, dv$$

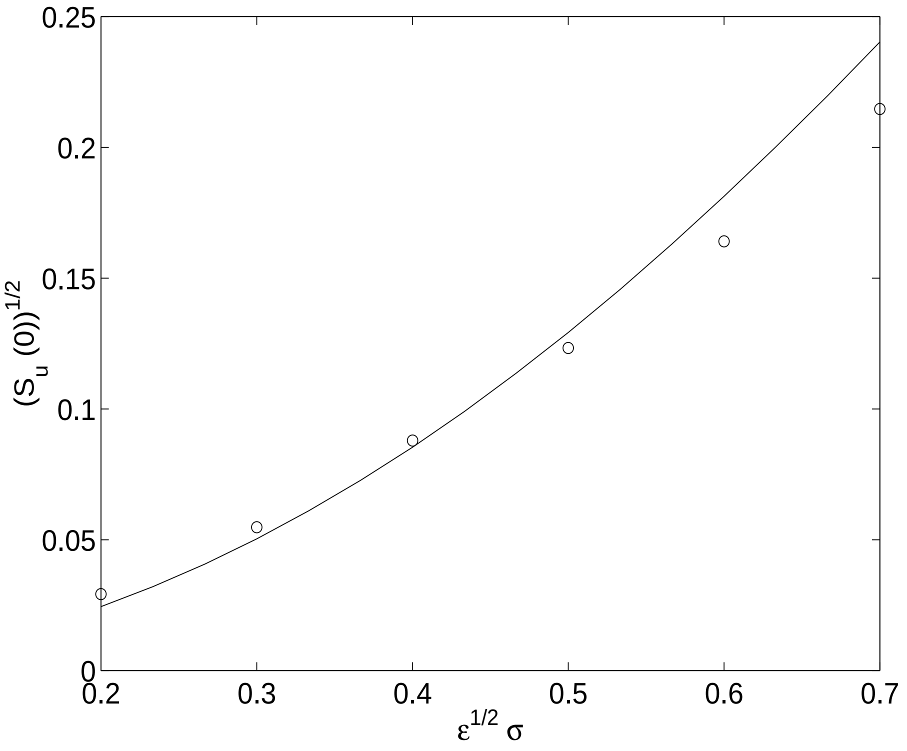

Figure 19. MC($\circ$).Only $n=1$ is taken in account.$S_w(\omega) \sim \omega^2$ as $\omega^2 \to 0$

Choreography and Stunts that have accompanied this presentation are of Michel Jean