VIRTUAL POWER, PSEUDOBALANCE, AND THE LAW OF ACTION AND REACTION

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On the method of virtual power in Continuum Mechanics

J. Mech. Mater. Struct. **4: 281-292, 2009**

On structural frame-indifference in Plasticity

submitted

Classical Continuum Mechanics

Postulates

1. **Choice of a system of external actions**

$$b\,,\,s$$

2. **Balance equations** (Euler 1776)

$$\int_\Omega b\,dV + \int_{\partial\Omega} s\,dA = 0\,, \quad \int_\Omega x\times b\,dV + \int_{\partial\Omega} x\times s\,dA = 0$$

3. **The stress principle for internal actions**
(Euler 1757, Cauchy 1823)

$$s = \hat{s}(x, \mathcal{S})$$

4. **The dependence on the normal** (Cauchy 1823)

$$\hat{s}(x,\mathcal{S}) = \tilde{s}(x,n)$$

Theorems

1. **The tetrahedron theorem** (Cauchy 1827)

$$\tilde{s}(x,n) = T(x)n$$

2. **Local equations of motion** (Cauchy 1827)

$$\mathrm{div} T + b = 0\,, \qquad T = T^T$$

3. **The equation of virtual power**
(Lagrange 1798, D'Alembert 1743)

$$\int_\Omega b \cdot v\, dV + \int_{\partial\Omega} s \cdot v\, dA = \int_\Omega T \cdot \nabla v\, dV$$

Boundary-value problems

1. **A constitutive equation**

$$T = \mathbb{C}\,\nabla u$$

2. **Weak formulation**

$$\int_{\Omega} \mathbb{C}\,\nabla u \cdot \nabla v \, dV = \int_{\Omega} b \cdot v \, dV + \int_{\partial\Omega} s \cdot v \, dA$$

Classical Continuum Mechanics based on a variational approach

The Method of Virtual Power (Germain, 1973)

The method of virtual power

1. **Postulate the expressions of the external and internal power:**

$$\mathcal{P}_e(\Pi, v) = \int_\Pi b \cdot v \, dV + \int_{\partial\Pi} s \cdot v \, dA$$

$$\mathcal{P}_i(\Pi, v) = \int_\Pi T \cdot \nabla v \, dV$$

2. **Postulate the equation of virtual power**

$$\mathcal{P}_e(\Pi, v) = \mathcal{P}_i(\Pi, v)$$

3. **Introduce the constitutive equation**

$$\int_\Omega \mathbb{C}\,\nabla u \cdot \nabla v \, dV = \int_\Omega b \cdot v \, dV + \int_{\partial\Omega} s \cdot v \, dA$$

The balance equations follow from (2), by the arbitrariness of v

Classical Continuum Mechanics based on an indifference axiom

(Noll, 1959)

1. **Choice of kinematics**

$$v \in \mathcal{V}$$

2. **External power as a continuous linear functional on $\mathcal{V}$**

$$\mathcal{P}_e(\Pi, v) \;=\; \int_\Pi b \cdot v \, dV + \int_{\partial\Pi} s \cdot v \, dA$$

3. **Invariance under changes of observer**

$$\mathcal{P}_e(\Pi, v) = \mathcal{P}_e(\Pi, v + c + \omega \times x)$$

4. **The balance equations <u>follow</u> from (3)**

$$\mathcal{P}_e(\Pi, c) = 0\,, \quad \mathcal{P}_e(\Pi, \omega \times x) = 0$$

5. **The internal power <u>follows</u> from the Divergence Theorem**

$$\mathcal{P}_e(\Pi, v) \;=\; \int_\Pi T \cdot \nabla v \, dV$$

no typo !!!

classical

- choice of external actions
- balance equations
- stress principle
- dependence on the normal[1]

virtual power

- choice of the external and internal powers[2]
- virtual power equation

indifference

- choice of the external power
- indifference of power

[1] The dependence on the normal is now a theorem (Noll 1959)

[2] Balance equations, stress principle, and dependence on the normal are implicit in the choice of the powers

Application to non-classical continua

In classical mechanics, a non-classical continuum is obtained by adding:

1. **New external actions**
   (couple stresses, microforces)
   (Cosserat 1909, Mindlin 1963, Eringen 1964, Capriz 1989, Gurtin 2004)
2. **New balance equations**
   (microforce balance)

Questions:

Are the new balance equations general laws of Nature?

Do they have an adequate experimental support?

An example:

The damage model of Frémond & Nedjar (1993)

The Virtual Power Approach

$$\mathcal{P}_e(\Pi, v, \nu) = \int_\Pi (b \cdot v + \beta\nu)\, dV + \int_{\partial\Pi} (s \cdot v + \sigma\nu)\, dA$$

$$\mathcal{P}_i(\Pi, v, \nu) = \int_\Pi (T \cdot \nabla v + \Sigma \cdot \nabla\nu)\, dV$$

ν the virtual variation of a scalar damage variable

β, σ the associated external actions

Σ the associated internal action (vector)

the equation of virtual power ...

$$
\begin{aligned}
0 &= \mathcal{P}_e(\Pi, v, \nu) - \mathcal{P}_i(\Pi, v, \nu) \\
&= \int_\Pi (b_i v_i + \beta\nu - T_{ij} v_{i,j} - \Sigma_i \nu_{,i})\, dV + \int_{\partial\Pi} (s_i v_i + \sigma\nu)\, dA \\
&= \int_\Pi \big((b_i + T_{ij,j})\, v_i + (\beta + \Sigma_{i,i})\, \nu \big)\, dV \\
&\qquad + \int_{\partial\Pi} \big((s_i - T_{ij} n_j)\, v_i + (\sigma - \Sigma_i n_i)\, \nu \big)\, dA
\end{aligned}
$$

... and the local equations

$$
b + \mathrm{div} T = 0\,, \qquad s = T\, n
$$

$$
\beta + \mathrm{div}\Sigma = 0\,, \qquad \sigma = \Sigma \cdot n
$$

Something, in this procedure, is not clear:

1. **What is the new balance equation?**

$$\int_{\Pi} \beta \, dV + \int_{\partial\Pi} \sigma \, dA = 0$$

   Is it a general law of nature like the law of balance of momentum?

2. **Does every new continuum generate a new general law ?**
3. **Is the choice of the internal and external powers arbitrary?**
4. **If not, which are the criteria for the choice?**

The Indifference approach

$$\mathcal{P}_e(\Pi, v, \nu) \;=\; \int_\Pi (b\cdot v + \beta\,\nu)\,dV + \int_{\partial\Pi} (s\cdot v + \sigma\,\nu)\,dA$$

indifference of power

$$\mathcal{P}_e(\Pi, c, \nu) = 0 \qquad \Rightarrow \qquad c\cdot\Big(\int_\Pi b\,dV + \int_{\partial\Pi} s\,dA\Big) = 0$$

$$\mathcal{P}_e(\Pi, \omega\times x, \nu) = 0 \qquad \Rightarrow \qquad \omega\cdot\Big(\int_\Pi x\times b\,dV + \int_{\partial\Pi} x\times s\,dA\Big) = 0$$

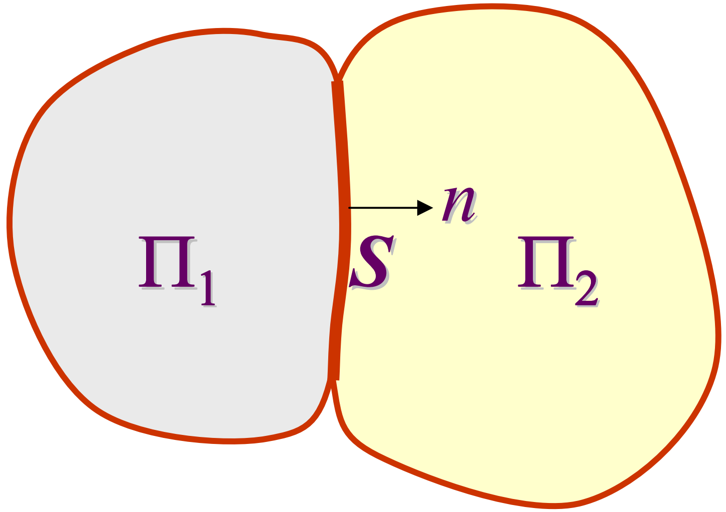
local relations

$$s = Tn\,, \qquad \mathrm{div}T + b = 0\,, \qquad T = T^T$$

no *internal* expression of power!

$$\mathcal{P}_e(\Pi, v, \nu) \;=\; \int_\Pi (T\cdot\nabla v + \beta\,\nu)\,dV + \int_{\partial\Pi} \sigma\,\nu\,dA$$

After this short introduction ...



Π_1, Π_2 **regions of R^N**

$S = \partial\Pi_1 \cap \partial\Pi_2$ **surface element**

(S, n) **oriented surface element**

$S = (S, n) \quad \Rightarrow \quad (S, -n) = -S$

Y **a finite dimensional inner product space**
(scalars, vectors, tensors)

A **Cauchy flux** is a map Q from the oriented surface elements to Y, additive on disjoint subsets (a ***measure***)

A Cauchy flux obeys the **law of action and reaction** if

$$Q(-S) = -Q(S)$$

Let M be the map from the regions Π to Y defined by

$$M(\Pi) = -Q(\partial\Pi)\,, \qquad \partial\Pi = (\partial\Pi, n_{ext})$$

Is M a measure?

$$\Pi_1 \cap \Pi_2 = \emptyset \quad \overset{?}{\Longrightarrow} \quad M(\Pi_1 \cup \Pi_2) = M(\Pi_1) + M(\Pi_2)$$

Let M be the map from the regions Π to Y defined by

$$M(\Pi) = -Q(\partial\Pi)\,, \qquad \partial\Pi = (\partial\Pi, n_{ext})$$

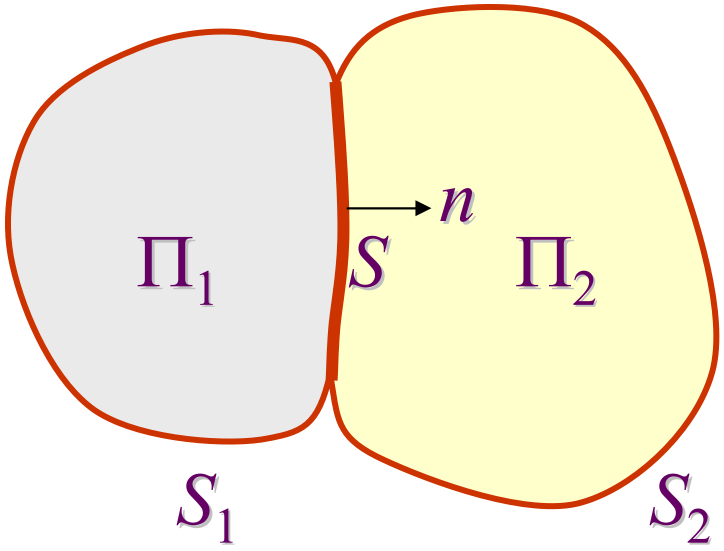
Is M a measure?

$$\Pi_1 \cap \Pi_2 = \emptyset \quad \overset{?}{\Longrightarrow} \quad M(\Pi_1 \cup \Pi_2) = M(\Pi_1) + M(\Pi_2)$$

Theorem (Noll 1973): **M is additive if and only if Q obeys the law of action and reaction**

sketch of proof

$$
\begin{aligned}
\partial\Pi_1 &= S_1 \cup S \\
\partial\Pi_2 &= S_2 \cup (-S) \\
\partial(\Pi_1 \cup \Pi_2) &= S_1 \cup S_2
\end{aligned}
$$



$$
\begin{aligned}
&M(\Pi_1) + M(\Pi_2) - M(\Pi_1 \cup \Pi_2) \\
&= -Q(\partial\Pi_1) - Q(\partial\Pi_2) + Q(\partial(\Pi \cup \Pi_2)) \\
&= -Q(S_1) - Q(S) - Q(S_2) - Q(-S) + Q(S_1) + Q(S_2) \\
&= -Q(S) - Q(-S)
\end{aligned}
$$

therefore, if the law of action and reaction holds, we have a relation between measures

$$M(\Pi) + Q(\partial\Pi) = 0$$

and, by the Radon-Nikodym theorem,

$$\int_{\Pi} m\, dV + M^s(\Pi) + \int_{\partial\Pi} s\, dA + Q^s(\partial\Pi) = 0$$

$Q^s(\partial\Pi) \neq 0$ **means that there are contact actions concentrated on lines or isolated points of** $\partial\Pi$

Assume that Q is absolutely continuous:

$$\int_{\Pi} m\,dV + M^s(\Pi) + \int_{\partial\Pi} s\,dA + \cancel{Q^s(\partial\Pi)} = 0$$

Question: does this imply that M is absolutely continuous?

Answers

1. (Gurtin & Martins, 1976) M **is absolutely continuous if Q is volume bounded:**

$$\exists k > 0 \quad : \quad |Q(\partial\Pi)| \leq kV(\Pi)$$

2. (Silhavy 1985) M **is absolutely continuous if and only if Q is weakly balanced:**

$$\exists f \in L^1(\Omega, Y) \quad : \quad |Q(\partial\Pi)| \leq \int_\Pi f\, dV$$

If Q satisfies the law of action and reaction and is weakly balanced, then

$$\int_{\Pi} m\, dV + \int_{\partial\Pi} s\, dA = 0$$

This looks like a balance equation but it is not. This is a representation of a Cauchy flux by a volume integral. It is possible only under ad hoc regularity assumptions.

We call this a pseudobalance equation

Let us go back to Frémond's model

$$\mathcal{P}_e(\Pi, v, \nu) \; = \; \int_\Pi (b \cdot v + \beta\,\nu)\,dV + \int_{\partial\Pi} (s \cdot v + \sigma\,\nu)\,dA$$

Assume that s and σ be surface densities of weakly balanced Cauchy fluxes which satisfy the law of action and reaction. Write the pseudobalance equations

$$\int_\Pi m\,dV + \int_{\partial\Pi} s\,dA = 0\,, \qquad \int_\Pi \mu\,dV + \int_{\partial\Pi} \sigma\,dA = 0$$

By the tetrahedron theorem,

$$s \; = \; T\,n\,, \qquad \sigma = \Sigma \cdot n$$

***T* is not anymore the Cauchy stress!**

by the Divergence Theorem,

$$\int_{\partial\Pi} s \cdot v \, dA = \int_{\partial\Pi} Tn \cdot v \, dA = \int_{\Pi} (T \cdot \nabla v + \mathrm{div} T \cdot v) \, dV$$

$$\int_{\partial\Pi} \sigma \, \nu \, dA = \int_{\partial\Pi} (\Sigma \cdot n) \, \nu \, dA = \int_{\Pi} (\Sigma \cdot \nabla \nu + (\mathrm{div}\Sigma) \, \nu) \, dV$$

so that

$$\mathcal{P}_e(\Pi, v, \nu) = \int_{\Pi} \Big((b + \mathrm{div} T) \cdot v + (\beta + \mathrm{div}\Sigma) \, \nu + T \cdot \nabla v + \Sigma \cdot \nabla \nu \Big) \, dV$$

Now postulate the indifference of power

$$\mathcal{P}_e(\Pi, c, \nu) = 0 \qquad \mathcal{P}_e(\Pi, \omega \times x, \nu) = 0$$

to get the genuine balance laws

$$\mathrm{div} T + b = 0 \, , \qquad T = T^T$$

The final expression of the power as a volume integral

$$\mathcal{P}_e(\Pi, v, \nu) = \int_{\Pi} \left(T \cdot \nabla v + \Sigma \cdot \nabla \nu + \left(\beta + \mathrm{div}\Sigma\right) \nu\right) dV$$

coincides with the expression of the internal power <u>postulated</u> in Frémond & Nedjar (1996)

A format for non-classical continua

External power

$$\mathcal{P}_e(\Pi, v, \nu^\alpha) = \int_\Pi \left(b \cdot v + \textstyle\sum_\alpha \beta^\alpha \nu^\alpha\right) dV + \int_{\partial\Pi} \left(s \cdot v + \textstyle\sum_\alpha \sigma^\alpha \nu^\alpha\right) dA$$

ν^α **virtual variations of a *state variable* or *order parameter*** (scalar, vector, tensor)

β^α, σ^α **the corresponding external actions**

Pseudobalance equations

$$\int_\Pi m \, dV + \int_{\partial\Pi} s \, dA = 0 \,, \qquad \int_\Pi \mu^\alpha \, dV + \int_{\partial\Pi} \sigma^\alpha \, dA = 0$$

From the tetrahedron theorem,

$$s = T\,n\,, \qquad \sigma^\alpha = \Sigma^\alpha \cdot n$$

$$\mathcal{P}_e(\Pi, v, \nu^\alpha) = \int_\Pi \Big((b + \mathrm{div} T)\cdot v + \textstyle\sum_\alpha (\beta^\alpha + \mathrm{div}\Sigma^\alpha)\,\nu^\alpha + T\cdot\nabla v + \sum_\alpha \Sigma^\alpha\cdot\nabla\nu^\alpha\Big)\, dV$$

If the indifference requirements are

$$\mathcal{P}_e(\Pi, c, \nu^\alpha) = 0\,, \qquad \mathcal{P}_e(\Pi, \omega\times x, \nu^\alpha) = 0$$

then

$$\mathcal{P}_e(\Pi, v, \nu^\alpha) = \int_\Pi \Big(T\cdot\nabla v + \textstyle\sum_\alpha \left(\Sigma^\alpha\cdot\nabla\nu^\alpha + (\beta^\alpha + \mathrm{div}\Sigma^\alpha)\,\nu^\alpha\right)\Big)\, dV$$

It is remarkable that the law of action and reaction, which in classical Continuum Mechanics is a trivial consequence of the balance of momentum, becomes a true postulate for a large class of non-classical continua

Second-gradient continua

External power

$$\mathcal{P}_e(\Pi, v) = \int_\Pi (b \cdot v + B \cdot \nabla v)\, dV + \int_{\partial\Pi} (s \cdot v + S \cdot \nabla v)\, dA$$

B, S **the external actions associated with** ∇v

(second-order tensors)

Pseudobalance equations

$$\int_\Pi m\, dV + \int_{\partial\Pi} s\, dA = 0\,, \qquad \int_\Pi M\, dV + \int_{\partial\Pi} S\, dA = 0$$

From the tetrahedron theorem,

$$s = T\,n\,, \qquad S = \mathsf{T}n\,, \qquad (S_{ij} = \mathsf{T}_{ijk}n_k)$$

$$\mathcal{P}_e(\Pi, v) = \int_\Pi \big((b + \mathrm{div}T)\cdot v + (B + \mathrm{div}\mathsf{T})\cdot\nabla v + T\cdot\nabla v + \mathsf{T}\cdot\nabla\nabla v\big)\,dV$$

$$\Big((\mathrm{div}\mathsf{T})_{ij} = \mathsf{T}_{ijk,k}\Big)$$

Indifference requirements

$$\mathcal{P}_e(\Pi, c) = 0 \quad \Rightarrow \quad c\cdot\int_\Pi (b + \mathrm{div}T)\,dV = 0$$

$$\mathcal{P}_e(\Pi, \omega\times x) = 0 \quad \Rightarrow \quad \omega\cdot\int_\Pi x\times(b + \mathrm{div}T)\,dV + W\cdot\int_\Pi (T + B + \mathrm{div}\mathsf{T})\,dV = 0$$

$$\Big(W = \nabla(\omega\times x)\,, \quad W = -W^T, \quad \nabla W = 0\Big)$$

Then,

$$b + \mathrm{div}T = 0\,, \qquad (T + B + \mathrm{div}\mathsf{T}) = (T + B + \mathrm{div}\mathsf{T})^T$$

The integral form of the power for a second-gradient continuum

(Germain 1973)

$$\mathcal{P}_e(\Pi, v) = \int_\Pi \big((T + B + \mathrm{div}\mathsf{T}) \cdot \nabla v + \mathsf{T} \cdot \nabla\nabla v \big)\, dV$$

Crystal Plasticity

(Rice 1971, Gurtin 2000)

External power

$$\mathcal{P}_e(\Pi, v, L) = \int_\Pi b \cdot v \, dV + \int_{\partial\Pi)} (s \cdot v + S \cdot L) \, dA$$

L **plastic strain rate** (second-order tensor)

S **the associated external action**

Pseudobalance equations

$$\int_\Pi m \, dV + \int_{\partial\Pi} s \, dA = 0 \,, \qquad \int_\Pi M \, dV + \int_{\partial\Pi} S \, dA = 0$$

From the tetrahedron theorem,

$$s = Tn, \qquad S = \mathsf{T}n$$

$$\mathcal{P}_e(\Pi, v, L) = \int_\Pi \big((b + \mathrm{div}T)\cdot v + \mathrm{div}\mathsf{T}\cdot L + T\cdot\nabla v + \mathsf{T}\cdot\nabla L\big)\,dV$$

Indifference requirements

$$\mathcal{P}_e(\Pi, c, L) = 0 \quad \Rightarrow \quad b + \mathrm{div}T = 0$$

$$\mathcal{P}_e(\Pi, \omega\times x, L) = 0 \quad \Rightarrow \quad T = T^T$$

The integral form of the power for crystal plasticity (Gurtin 2000)

$$\mathcal{P}_e(\Pi, v, L) = \int_\Pi (T\cdot\nabla v + \mathrm{div}\mathsf{T}\cdot L + \mathsf{T}\cdot\nabla L)\,dV$$

Perspectives

1. **Extensions to surface measures $\mathcal{Q}$ which are not absolutely continuous** (e.g., edge forces)

   Noll & Virga 1990

   Dell'Isola & Seppecher 1995, 1997

   Podio Guidugli 2004

   Degiovanni et al 2006

2. Extensions to surface measures Q which are not weakly balanced

(e.g., singular body forces)

Degiovanni et al 1999

Silhavy 2005, 2008

Lucchesi et al 2007

Schuricht 2007

THE END

In the variational approach, the criteria for choosing the expressions of the two powers are not clear

~~The two expressions are not really arbitrary, since it is tacitly agreed that the outcoming balance equations should not contradict the classical balance equations~~

~~This sort of pre-selection conflicts with the status of a postulate attributed to the choice of the expressions of the two powers~~