On the Route to a Variational Formulation for Impacts

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with many thanks to my senior researchers

Remco Leine
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for both developing the theory and the transparencies

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I. Objective

- Principle of Hamilton for perfect bilateral constraints

$$J(\vec{q}) = \int_I L(\vec{q}, \dot{\vec{q}})\, dt \rightarrow \text{stationary} \quad + \quad \text{boundary conditions}$$

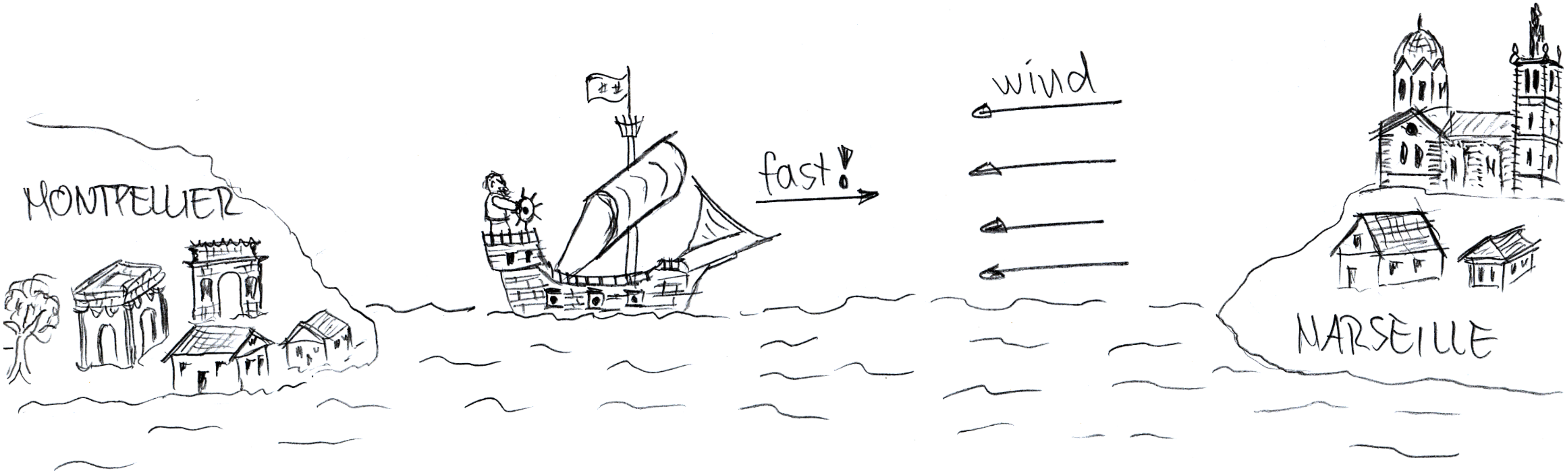
$$\Rightarrow \delta J = 0 \quad \forall \delta\vec{q} \qquad \text{variational equality}$$

- Here: Aim to derive the (?) principle of Hamilton
  for perfect unilateral constraints
  in the form of a variational inequality

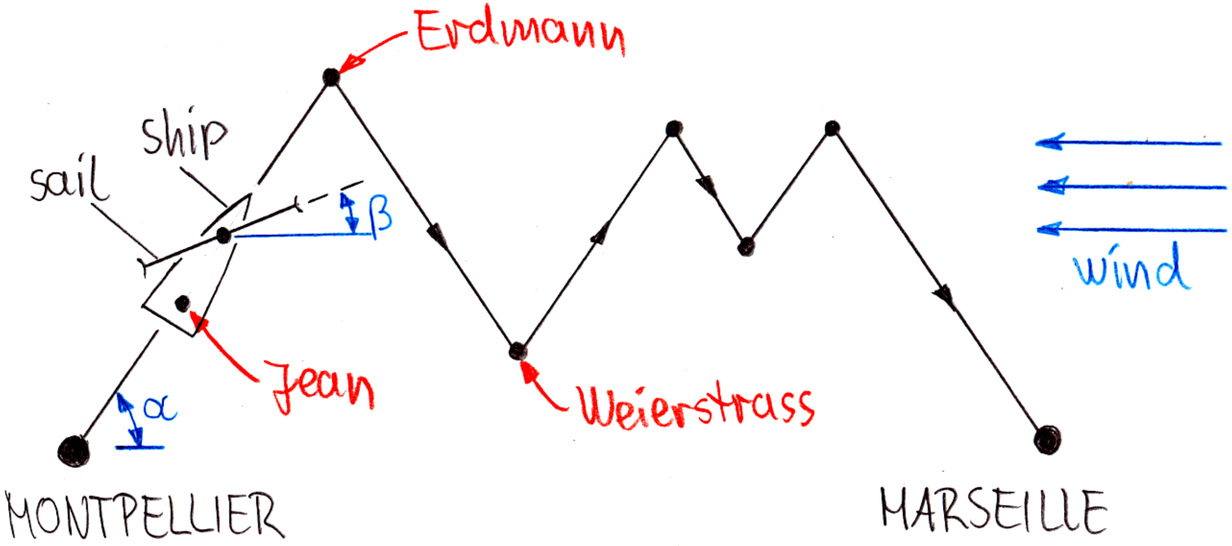
! Solutions to variational problems might be non-smooth

• Example:

Sailing against the wind



- Sailing against the wind



- History

1985: determine (α, β) in mathematics exercise

1994: 2nd CMIS, Marseille / Carry-le-Rouet

P.D. Pana: "Variational Principles for Contact Problems Including Impact Phenomena"

2006: Leine / Aeberhard: find of Panos/Glocker paper

II. Strong and Weak Variations

- Notation: Consider family of curves $\hat{f}(\varepsilon, t)$. We denote:

$$f(t) := \hat{f}(0,t) \quad \text{"solution"} \quad ; \quad \delta f(t) := \left.\frac{\partial \hat{f}(\varepsilon,t)}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \delta\varepsilon \quad \text{1st variation}$$

- Function space and norms: Let

$$\mathcal{Y}(I, \mathbb{R}^n) = \{ \vec{q} \in AC(I, \mathbb{R}^n) \text{ with } \dot{\vec{q}} \in SBV(I, \mathbb{R}^n)$$

and introduce two distances on $\mathcal{Y}(I, \mathbb{R}^n)$:

strong distance: $\|\vec{q}^* - \vec{q}\|_0 := d_0(\vec{q}^*, \vec{q}) := \sum_{i=1}^{n} \sup_{t\in I} |q_i^*(t) - q_i(t)|$

weak distance: $\|\vec{q}^* - \vec{q}\|_1 := d_1(\vec{q}^*, \vec{q}) := \sum_{i=1}^{n} \sup_{t\in I} |q_i^*(t) - q_i(t)| + \operatorname{ess\,sup}_{t\in I} |\dot{q}_i^*(t) - \dot{q}_i(t)|$

essential supremum does not look at impact time-events, for which $\dot{\vec{q}}(t)$ is not defined!

$\Rightarrow \|\vec{q}^* - \vec{q}\|_0 \leq \|\vec{q}^* - \vec{q}\|_1$

• <u>Definition:</u> <u>Weak</u> family of test functions:

$$\lim_{\varepsilon \downarrow 0} \| \hat{\vec{q}}(\varepsilon,t) - \vec{q}(t) \|_1 = 0 \qquad \left(\Rightarrow \lim_{\varepsilon \downarrow 0} \| \hat{\vec{q}}(\varepsilon,t) - \vec{q}(t) \|_0 = 0 \right)$$

weak norm

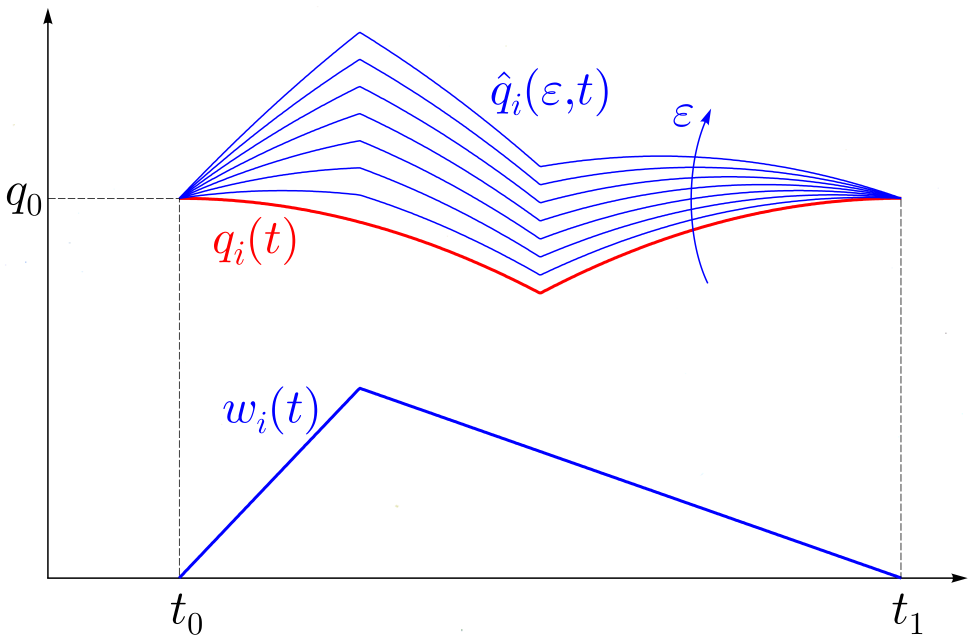
strong norm

• Example: Let $\vec{w}(t)$ be AC and

$$\hat{\vec{q}}(\varepsilon,t) = \vec{q}(t) + \varepsilon\, \vec{w}(t)$$

$$\Rightarrow \lim_{\varepsilon \downarrow 0} \| \hat{\vec{q}}(\varepsilon,t) - \vec{q}(t) \|_1 = \lim_{\varepsilon \downarrow 0} \| \varepsilon\, \vec{w}(t) \|_1$$

$$= \lim_{\varepsilon \downarrow 0} \varepsilon \left(\sum_i \sup_{t \in I} |w_i(t)| + \operatorname*{ess\,sup}_{t \in I} |\dot{w}_i(t)| \right) = 0$$



• "Weak" variation

$$\delta \vec{q}(t) = \left. \frac{\partial \hat{\vec{q}}(\varepsilon,t)}{\partial \varepsilon} \right|_{\varepsilon=0} \delta\varepsilon = \vec{w}(t)\, \delta\varepsilon$$

<u>continuous!</u>

• **Definition:** Strong family of test functions: ! Larger class of test functions

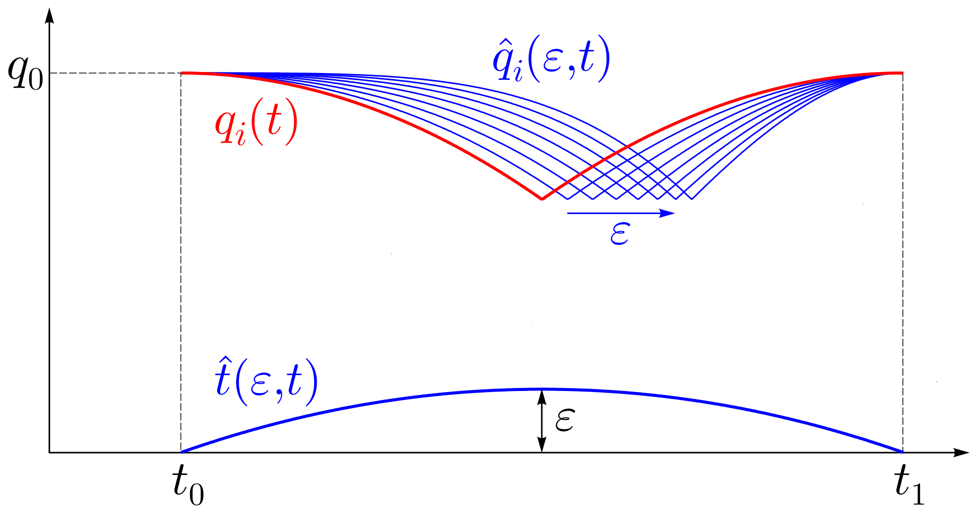
$$\lim_{\varepsilon \downarrow 0} \|\hat{\vec{q}}(\varepsilon,t) - \vec{q}(t)\|_{\underbrace{0}_{\text{strong norm}}} = 0 \qquad \Big(\lim_{\varepsilon \downarrow 0} \|\hat{\vec{q}}(\varepsilon,t) - \vec{q}(t)\|_{\underbrace{1}_{\text{weak norm}}} \neq 0 \text{ might happen!}\Big)$$

• Example: Let $\hat{t}(\varepsilon,t)$ be smooth with $\hat{t}(0,t)=0 \ \forall t$ and

$$\hat{\vec{q}}(\varepsilon,t) = \vec{q}(t - \hat{t}(\varepsilon,t))$$

$$\Rightarrow \lim_{\varepsilon \downarrow 0} \|\hat{\vec{q}}(\varepsilon,t) - \vec{q}(t)\|_0$$

$$= \lim_{\varepsilon \downarrow 0} \sum_i \sup_{t \in I} |q_i(t - \underbrace{\hat{t}(\varepsilon,t)}_{\text{for } \varepsilon \downarrow 0 \ \to 0}) - q_i(t)| = 0$$



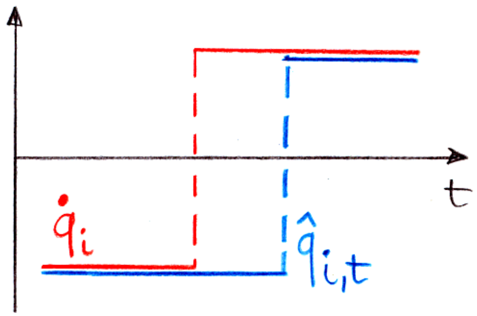
But: $\lim_{\varepsilon \downarrow 0} \|\hat{\vec{q}}(\varepsilon,t) - \vec{q}(t)\|_1$

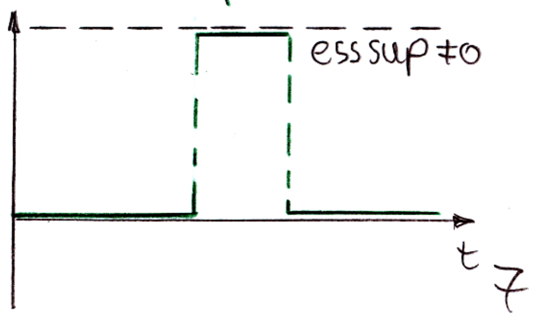
$$= \ldots + \lim_{\varepsilon \downarrow 0} \operatorname*{ess\,sup}_{t \in I} \underbrace{|\hat{q}_{i,t}(\varepsilon,t) - \dot{q}_i(t)|}_{} \neq 0$$

• "Strong" variation

$$\delta\vec{q}(t) = \frac{\partial \hat{\vec{q}}(\varepsilon,t)}{\partial \varepsilon}\Big|_{\varepsilon=0} \delta\varepsilon = -\dot{\vec{q}}(t)\, \hat{t}_{\varepsilon}(0,t)\, \delta\varepsilon$$

↳ discontinuous!





III. Unilateral Constraints

- Let $C \subseteq \mathbb{R}^n$ be the constraint set of $\vec{q}(t)$, i.e. — tang. regularity assumed

$$\vec{q}(t) \in C \quad \text{for each } t \in I$$

- This induces for the function $\vec{q}$ the set

$$\mathcal{C} := \{\vec{q} \mid \vec{q}(t) \in C \;\; \forall t \in I\}$$

- Consequences on virtual displacements $\delta\vec{q}$:

$$\hat{\vec{q}}(0+\delta\varepsilon, t) = \hat{\vec{q}}(0,t) + \underbrace{\hat{\vec{q}}_{,\varepsilon}(0,t)\,\delta\varepsilon}_{\delta\vec{q}} + \dots$$

If $\hat{\vec{q}}(\varepsilon) \in \mathcal{C} \;\; \forall \varepsilon \geq 0$, then it holds that

$$\delta\vec{q} \in \mathcal{T}_{\mathcal{C}}(\vec{q}) \Leftrightarrow \delta\vec{q}(t) \in T_C(\vec{q}(t)) \;\; \forall t$$

tangent cone

- Special case of weak test functions — $\hat{\vec{q}}(\varepsilon) = \vec{q} + \varepsilon\vec{w}\,;\;\; \delta\vec{q} = \vec{w}\,\delta\varepsilon$

$$\vec{w} \in \mathcal{T}_{\mathcal{C}}(\vec{q})$$

IV. Local Minima

- $\vec{q} \in \mathcal{C}$ is said to provide a functional J with a local minimum on $\mathcal{C}$ in the norm $\|\cdot\|_\alpha$ if $\exists\, r > 0$ such that

$$J(\vec{q}) \leq J(\vec{q}^*) \quad \forall \vec{q}^* \in B_r^\alpha \qquad \text{with } B_r^\alpha(\vec{q}) = \{\vec{q}^* \in \mathcal{C} \mid \|\vec{q}^* - \vec{q}\|_\alpha < r\}$$

- Applied to families of curves, this yields for $\vec{q} \in \mathcal{C}$

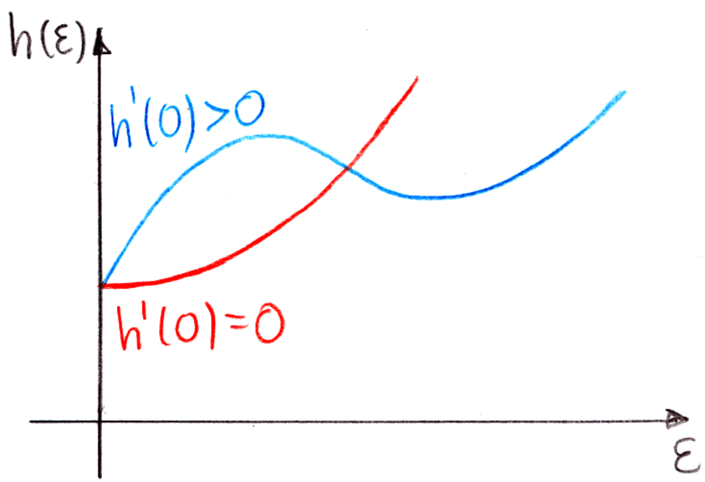
$$J(\hat{\vec{q}}(\varepsilon)) - J(\vec{q}) \geq 0 \qquad \forall \varepsilon \in [0, \varepsilon^*) \quad \forall \hat{\vec{q}}(\varepsilon) \in \mathcal{C}$$

$$\Rightarrow \boxed{\delta J := J_\varepsilon \delta\varepsilon = \delta\varepsilon \cdot \lim_{\varepsilon \downarrow 0} \frac{J(\hat{\vec{q}}(\varepsilon)) - J(\vec{q})}{\varepsilon} \geq 0 \qquad \forall \delta\vec{q} \in \tilde{T}_e(\vec{q})}$$

substationarity condition

- In other words, the function $h(\varepsilon) = J(\hat{\vec{q}}(\varepsilon))$ attains a local minimum at $\varepsilon = 0$, i.e. $h(0) = J(\vec{q})$.

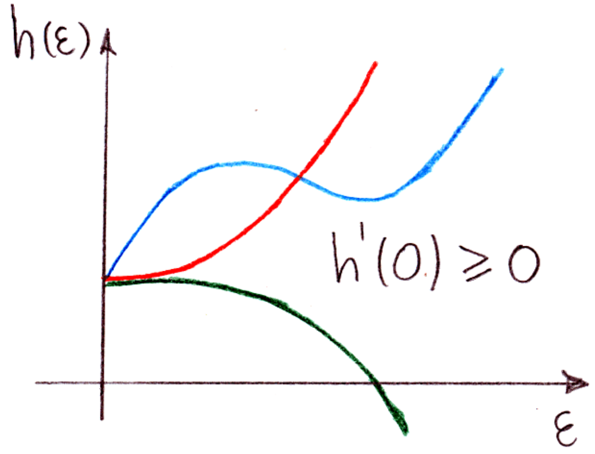
Necessary condition: $h'(0) \geq 0$



V. Local Extrema

- $\vec{q} \in \mathcal{C}$ is said to provide J with a local extremum on $\mathcal{C}$ if the substationarity condition applies:

$$\delta J = \delta\varepsilon \cdot \lim_{\varepsilon \downarrow 0} \frac{J(\hat{\vec{q}}(\varepsilon)) - J(\vec{q})}{\varepsilon} \geq 0 \quad \forall\, \delta\vec{q} \in \tilde{J}_e(\vec{q})$$



- Special case of weak test functions

$$\hat{\vec{q}}(\varepsilon) = \vec{q} + \varepsilon\vec{w}\,; \quad \delta\vec{q} = \vec{w}\,\delta\varepsilon$$

$$\Rightarrow \delta J = \delta\varepsilon \cdot \underbrace{\lim_{\varepsilon \downarrow 0} \frac{J(\vec{q} + \varepsilon\vec{w}) - J(\vec{q})}{\varepsilon}}_{\text{(one-sided) Gâteaux derivative } dJ(\vec{q};\vec{w})} \geq 0 \quad \forall\, \vec{w}\,\delta\varepsilon \in \tilde{J}_e(\vec{q})$$

$$\Rightarrow \underline{\underline{dJ(\vec{q};\vec{w}) \geq 0 \quad \forall\, \vec{w} \in \tilde{J}_e(\vec{q})}}$$

weak substationarity condition

- <u>Summary:</u>

(i) A strong extremal $\vec{q}$ of J on $\mathcal{C}$ fulfills

$$\delta J \geq 0 \quad \forall\, \delta\vec{q} \in \tilde{\mathcal{J}}_{\mathcal{C}}(\vec{q})$$

(ii) A weak extremal $\vec{q}$ of J on $\mathcal{C}$ fulfills just

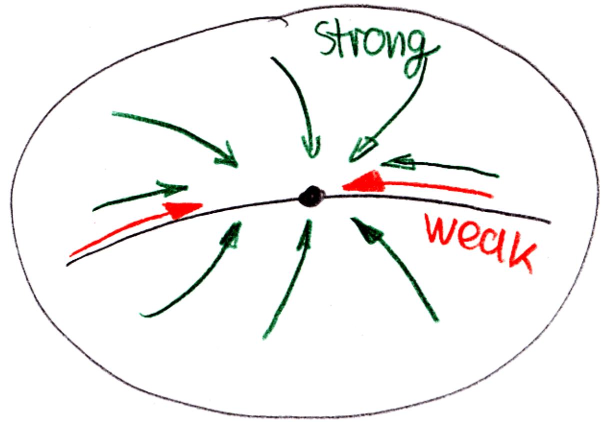
$$dJ(\vec{q};\vec{w}) \geq 0 \quad \forall\, \vec{w} \in \tilde{\mathcal{J}}_{\mathcal{C}}(\vec{q})$$

- <u>Corollary:</u>

A strong loc. extremal is always a weak loc. extremal but not vice versa,

<u>because</u>

a strong loc. extremal is compared with a larger class of test functions than a weak local extremal



VI. Strong Virtual Displacements

- Consider test functions of the form

$$\hat{\vec{q}}(\varepsilon,t) = \vec{q}(t-\hat{t}(\varepsilon,t)) + \varepsilon\vec{w}(t-\hat{t}(\varepsilon,t)) \qquad (*)$$

such that
$$\begin{cases} \hat{\vec{q}}(0,t) = \vec{q}(t) \quad \forall t & \Rightarrow \hat{\vec{q}}_t(0,t) = \dot{\vec{q}}(t) \\ \quad\Downarrow & \\ \hat{t}(0,t) = 0 \quad \forall t & \Rightarrow \hat{t}_t(0,t) = 0 \end{cases}$$

variation parameter: $\delta\varepsilon := \varepsilon - \overset{=0}{\varepsilon_0} = \varepsilon$ small!

virtual time shift: $\delta t := \hat{t}_\varepsilon(0,t)\,\delta\varepsilon$

virtual value shift: $\vec{w}\,\delta\varepsilon$

- Virtual displacements

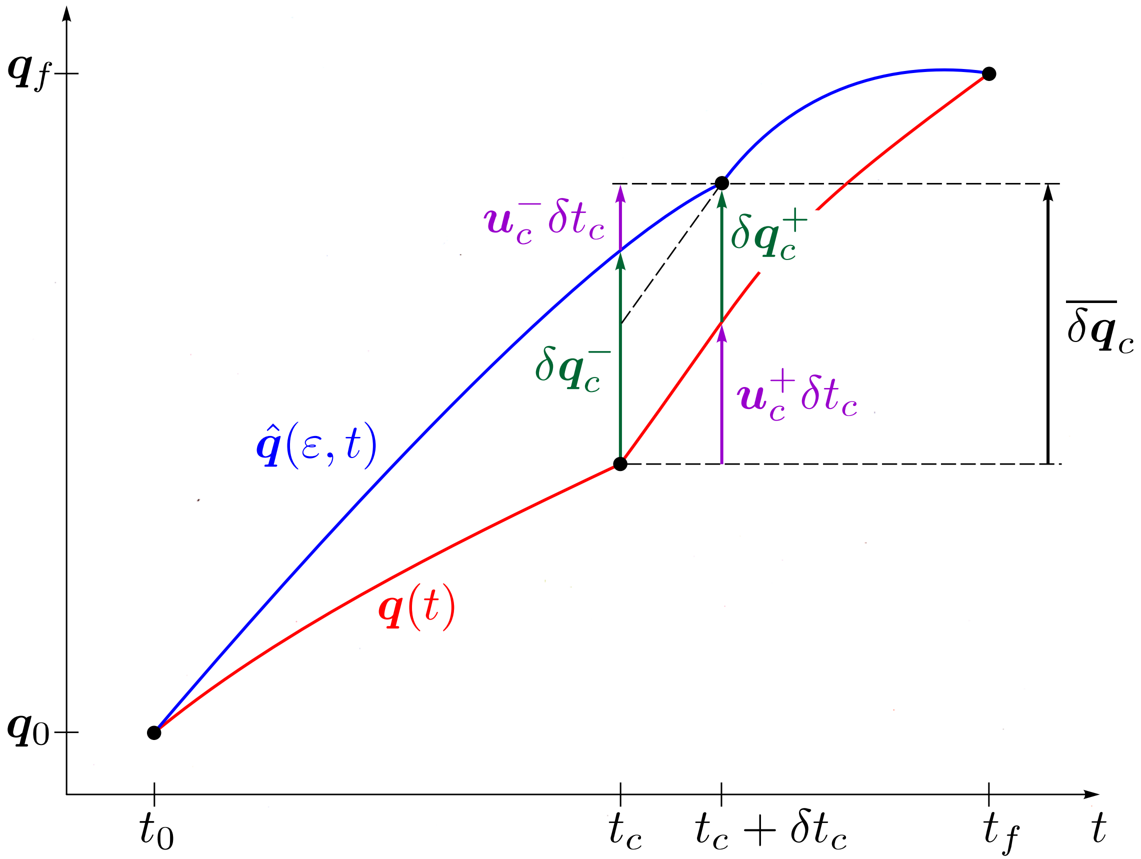
$$\delta\vec{q}(t) := \left.\frac{\partial\hat{\vec{q}}(\varepsilon,t)}{\partial\varepsilon}\right|_{\varepsilon=0}\delta\varepsilon \overset{(*)}{=} -\dot{\vec{q}}(t)\,\hat{t}_\varepsilon\,\delta\varepsilon + \vec{w}(t)\,\delta\varepsilon = -\vec{u}(t)\,\delta t + \vec{w}(t)\,\delta\varepsilon$$

Velocity $\vec{u}(t) \overset{a.e.}{=} \dot{\vec{q}}(t)$ is $LBV(I,\mathbb{R}^n)$ and thus discontinuous

$$\Rightarrow \underline{\underline{\delta\vec{q}^{\pm}(t) = -\vec{u}^{\pm}(t)\,\delta t + \vec{w}(t)\,\delta\varepsilon}}$$

strong virt. displacement is discontinuous in time!

$$\delta\vec{q}^{\pm}(t) = -\vec{u}^{\pm}(t)\,\delta t + \vec{w}(t)\,\delta\varepsilon$$



$$\boxed{\begin{aligned}\overline{\delta\vec{q}}_c &= \delta\vec{q}_c^{\,-} + \vec{u}_c^{\,-}\,\delta t_c \\ &= \delta\vec{q}_c^{\,+} + \vec{u}_c^{\,+}\,\delta t_c\end{aligned}}$$

- Introduce

  $\overline{\delta\vec{q}}(t) := \vec{w}(t)\,\delta\varepsilon$ virt. value shift

  $\overset{\text{a.e.}}{=} \delta\vec{q}(t) + \vec{u}(t)\,\delta t$ continuous in t!

- We can interprete $\delta\vec{q}$ as

  $\delta\vec{q}(t) \approx \hat{\vec{q}}(\varepsilon_0 + \delta\varepsilon, t) - \vec{q}(t)$

  fixed time

- And interprete $\overline{\delta\vec{q}}$ as

  $$\overline{\delta\vec{q}}(t) \approx \hat{\vec{q}}(\varepsilon_0 + \delta\varepsilon, t + \delta t) - \vec{q}(t)$$

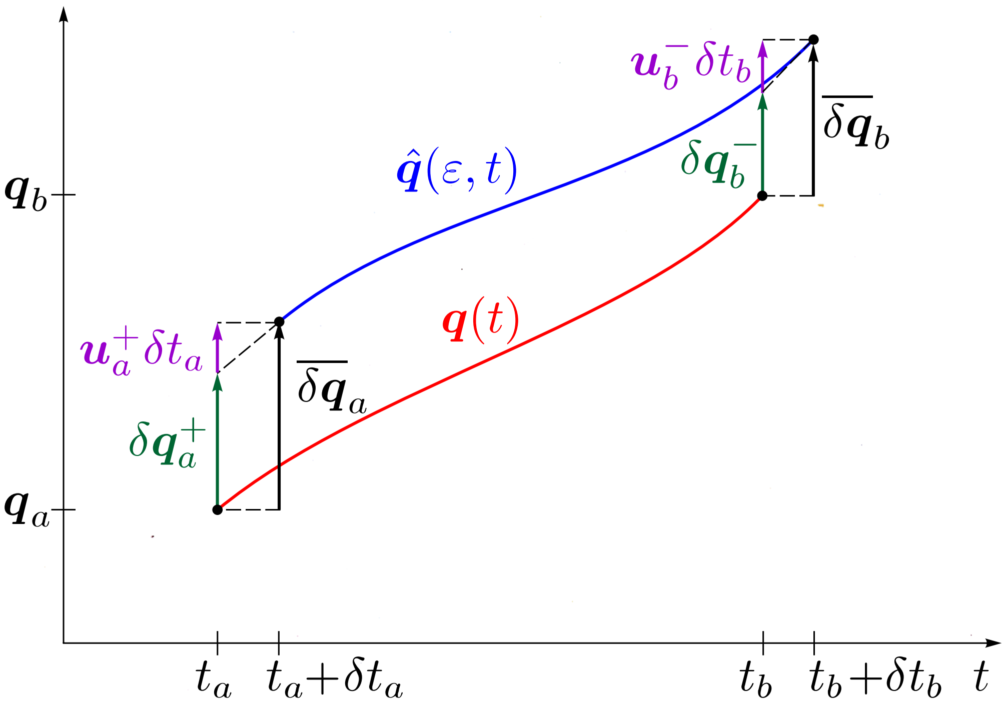
  $$= \hat{\vec{q}}(\varepsilon_0, t) + \hat{\vec{q}}_\varepsilon\,\delta\varepsilon + \hat{\vec{q}}_t\,\delta t - \vec{q}(t)$$

  $$\hat{\vec{q}}_t = \dot{\vec{q}}\,(1 - \underbrace{\hat{t}_t}_{0=}) = \dot{\vec{q}}$$

  $$= \delta\vec{q} + \vec{u}\,\delta t = \vec{w}\,\delta\varepsilon$$

  varied time

VII. Variation with Free Boundaries:



Let $\vec{q}(t)$ be smooth on $[t_a, t_b]$ and

$$J(\vec{q}) = \int_{t_a}^{t_b} L(\vec{q}, \dot{\vec{q}})\, dt$$

$$\delta J = \int_{t_a}^{t_b} \delta L\, dt + \left[L\, \delta t\right]_{t \downarrow t_a}^{t \uparrow t_b}$$

$$= \int_{t_a}^{t_b} \left(\frac{\partial L}{\partial \vec{q}}\, \delta \vec{q} + \frac{\partial L}{\partial \vec{u}}\, \delta \vec{u}\right) dt + \left[L\, \delta t\right]_{t \downarrow t_a}^{t \uparrow t_b}$$

$$= \int_{t_a}^{t_b} \left(\frac{\partial L}{\partial \vec{q}} - \frac{d}{dt}\frac{\partial L}{\partial \vec{u}}\right) \delta \vec{q}\, dt + \left[\frac{\partial L}{\partial \vec{u}}\, \delta \vec{q} + L\, \delta t\right]_{t \downarrow t_a}^{t \uparrow t_b}$$

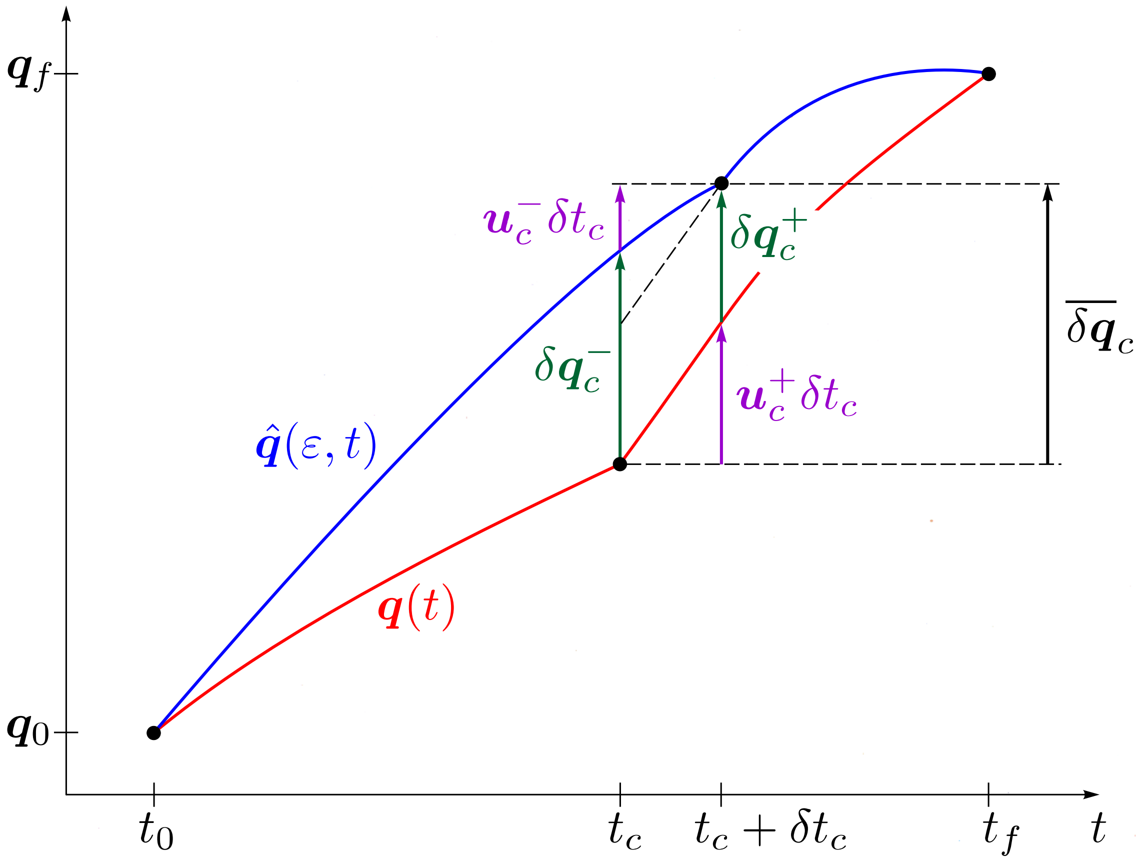
$$\vec{p}(\vec{q}, \vec{u}) := \left(\frac{\partial L}{\partial \vec{u}}\right)^T \qquad H(\vec{q}, \vec{u}) := \frac{\partial L}{\partial \vec{u}}\, \vec{u} - L$$

- Boundary terms:

$$\lim_{t \downarrow t_a} \left(\frac{\partial L}{\partial \vec{u}}\, \delta \vec{q} + L\, \delta t\right) = \vec{p}(\vec{q}_a, \vec{u}_a^+)^T \underbrace{\left(\delta \vec{q}_a^+ + \vec{u}_a^+\, \delta t_a\right)}_{\overline{\delta \vec{q}_a}} - H(\vec{q}_a, \vec{u}_a^+)\, \delta t_a$$

$$\lim_{t \uparrow t_b} \left(\frac{\partial L}{\partial \vec{u}}\, \delta \vec{q} + L\, \delta t\right) = \vec{p}(\vec{q}_b, \vec{u}_b^-)^T \underbrace{\left(\delta \vec{q}_b^- + \vec{u}_b^-\, \delta t_b\right)}_{\overline{\delta \vec{q}_b}} - H(\vec{q}_b, \vec{u}_b^-)\, \delta t_b$$

VIII. Variation with Corner



- We have:

$$\delta \vec{q}_c^{\pm} \in T_c(\vec{q}(t_c))$$

$$-\vec{u}_c^{-} \in T_c(\vec{q}(t_c))$$

$$+\vec{u}_c^{+} \in T_c(\vec{q}(t_c))$$

- We obtain:

$$\underline{\underline{\overline{\delta \vec{q}_c}}} = \delta \vec{q}_c^{-} + \vec{u}_c^{-} \delta t_c$$

$$= \delta \vec{q}_c^{+} + \vec{u}_c^{+} \delta t_c \;\; \underline{\underline{\in T_c(\vec{q}(t_c))}}$$

$\Rightarrow \forall \delta \vec{q}_c^{\pm} \in T_c(\vec{q}(t_c)), \; \forall \delta t_c \in \mathbb{R}$
induces $\forall \overline{\delta \vec{q}_c} \in T_c(\vec{q}(t_c))$

- Variation of the action integral

$$\delta J = \int_I \underbrace{\left(\frac{\partial L}{\partial \vec{q}} - \frac{d}{dt} \frac{\partial L}{\partial \vec{u}} \right) \delta \vec{q} \, dt}_{\text{Euler-Lagrange}} - \underbrace{\underbrace{(\vec{p}_c^{\,+} - \vec{p}_c^{\,-})^T \overline{\delta \vec{q}_c}}_{\text{kinetic term}} + \underbrace{(H_c^+ - H_c^-) \delta t_c}_{\text{energetic term}}}_{\text{depend on impact law!}}$$

$$-\delta J = \int_I \left(\frac{d}{dt}\frac{\partial L}{\partial \vec{u}} - \frac{\partial L}{\partial \vec{q}}\right)\delta\vec{q}\, dt + (\vec{p}_c^{\,+} - \vec{p}_c^{\,-})^T \overline{\delta\vec{q}_c} - (H_c^+ - H_c^-)\,\delta t_c \geqslant 0 \quad \forall \delta\vec{q} \in T_e(\vec{q}) \quad \forall \delta t_c \in \mathbb{R}$$

- Euler-Lagrange term: (a.e.)

$$-\underbrace{\left(\frac{d}{dt}\frac{\partial L}{\partial \vec{u}} - \frac{\partial L}{\partial \vec{q}}\right)}_{=:\vec{f}^T}\delta\vec{q} \leqslant 0 \quad \forall \delta\vec{q} \in T_e(\vec{q}) \quad \Leftrightarrow \quad -\vec{f} \in N_e(\vec{q})$$

contact & take-off

- Kinetic term (1^{st} generalized Weierstrass-Erdmann condition)

$$-\underbrace{(\vec{p}_c^{\,+} - \vec{p}_c^{\,-})^T}_{=:\vec{R}_c^T} \overline{\delta\vec{q}_c} \leqslant 0 \quad \forall \overline{\delta\vec{q}_c} \in T_c(\vec{q}(t_c)) \quad \Leftrightarrow \quad -\vec{R}_c \in N_c(\vec{q}(t_c))$$

impulsive force

- Energetic term (2^{nd} Weierstrass-Erdmann condition)

$$(H_c^+ - H_c^-)\,\delta t_c \leqslant 0 \quad \forall \delta t_c \in \mathbb{R} \quad \Leftrightarrow \quad H_c^+ = H_c^-$$

preservation of energy (completely elastic impact)

vanishes if either

- strong variations are taken and elastic impact law, or
- weak variations are taken for which $\delta t_c \equiv 0$

- Two principles of Hamilton

(i) strong principle of Hamilton

$$\boxed{-\delta J \geqslant 0 \qquad \forall\, \delta\vec{q} \in \mathcal{T}_e(\vec{q})}$$

valid for perfect unilateral constraints
and completely elastic impact laws

(ii) weak principle of Hamilton

$$\boxed{-dJ(\vec{q}\,;\,\delta\vec{q}) \geqslant 0 \quad \forall\, \delta\vec{q} = \vec{w}\,\delta\varepsilon \in \mathcal{T}_e(\vec{q})}$$

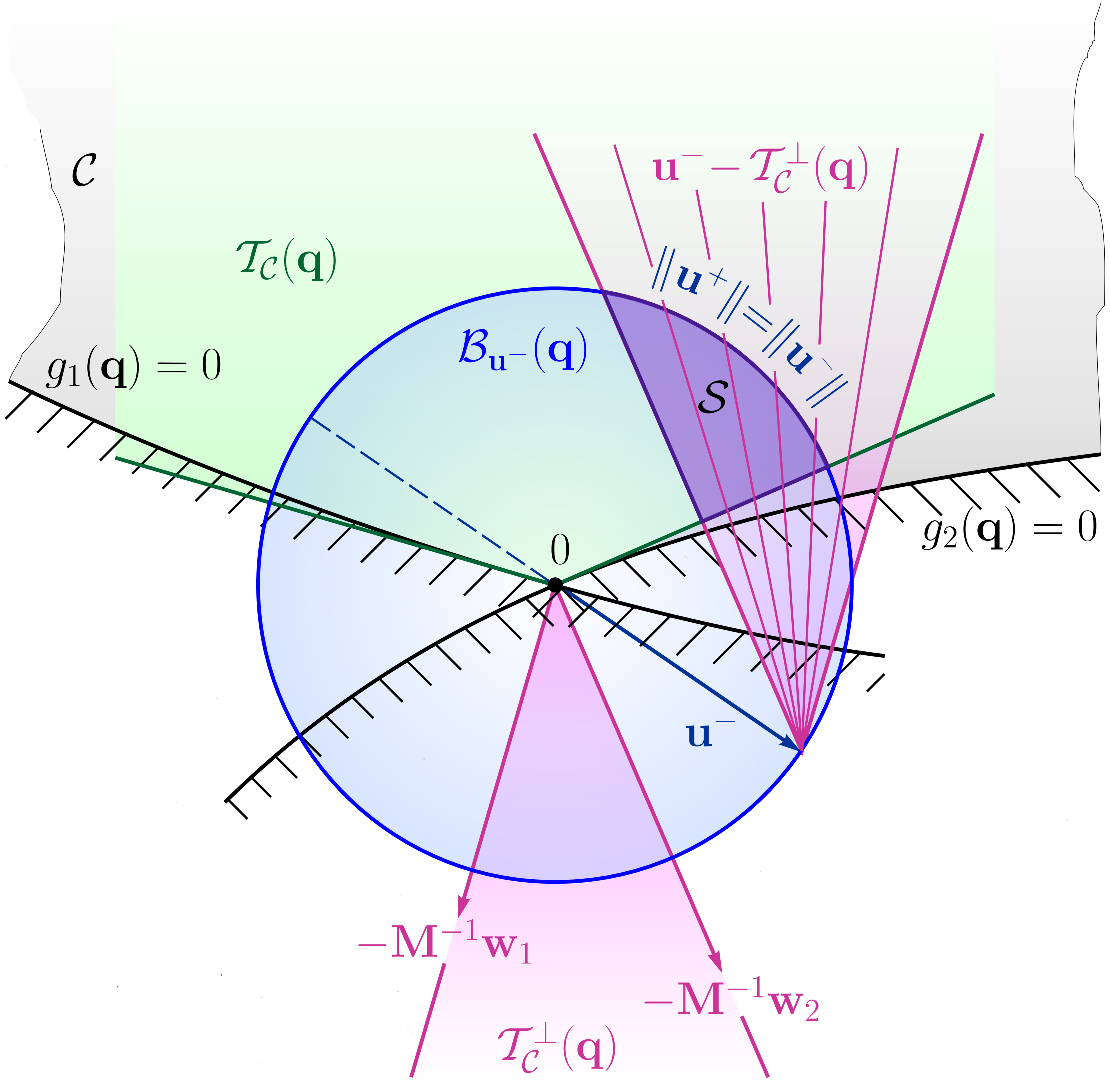
valid for perfect unilateral constraints

! no further condition on the impact law

IX. Conclusion

We are able to generalize the principle of Hamilton to perfect unilateral constraints, but we need to distinguish between weak and strong variations.

From the strong principle of Hamilton follows an energy preserving type of impact law, but this does not specify the impact law itself!



X. Outlook

- Derive Hamilton's principle by starting from the measure inequality?

$$\bar{\delta}\vec{q}^{T}\left(M\,d\vec{u}-\vec{h}\,dt-d\vec{R}\right)\geq 0 \quad \forall ??$$

??

- Where is the $\frac{1}{2}(\vec{u}^{+}+\vec{u}^{-})$?

$$\overline{\delta\vec{q}} = \delta\vec{q}^{\pm}+\vec{u}^{\pm}\,\delta t \qquad \bar{\delta}\ldots \rightsquigarrow \tfrac{1}{2}(\vec{u}^{+}+\vec{u}^{-})\,?$$

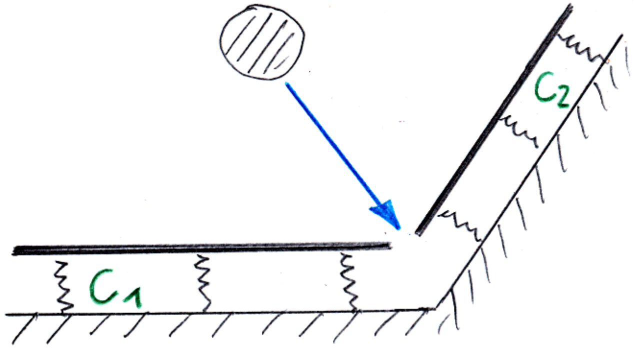
Reason: $\frac{1}{2}(\vec{u}^{+}+\vec{u}^{-})\cdot M(\vec{u}^{+}-\vec{u}^{-}) = T^{+}-T^{-}$

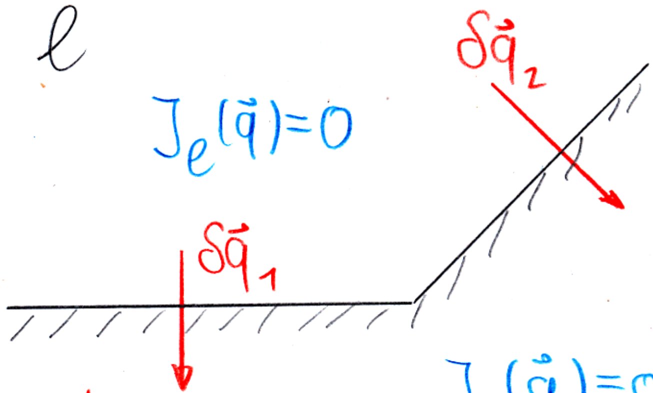
- Unilateral constraints by indicator potential?

$$V = V_{smooth}+V_{support}+\mathcal{I}_{e}(\vec{q})$$

$$\frac{\partial V}{\partial \vec{q}}\,\delta\vec{q} \rightsquigarrow \frac{\partial V_{smooth}}{\partial \vec{q}}\,\delta\vec{q} + V^{\uparrow}(\vec{q};\delta\vec{q}) + \mathcal{I}_{e}^{\uparrow}(\vec{q};\delta\vec{q}) \qquad \text{sub-derivative}$$

- Where is the impact law?





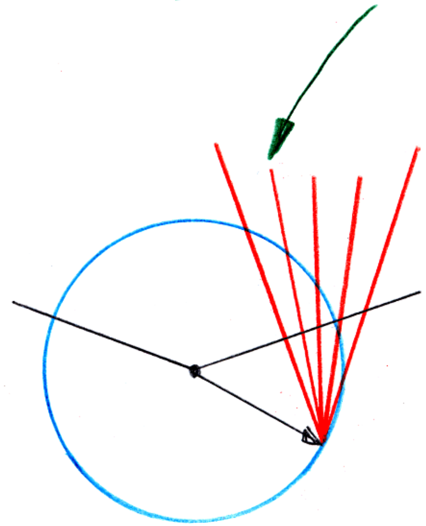
$J_e(\vec{q})=0$

$J_e^{\uparrow}(\vec{q},\delta\vec{q}_2)=\infty$

$J_e^{\uparrow}(\vec{q},\delta\vec{q}_1)=\infty$

$J_c(\vec{q})=\infty$

$$\underline{\underline{\frac{J_e^{\uparrow}(\vec{q},\delta\vec{q}_2)}{J_e^{\uparrow}(\vec{q},\delta\vec{q}_1)}=\frac{c_2}{c_1}}}$$



- Infinitely stiff springs?

$m\ddot{x}+cx=0$ $\qquad c\neq 0 \rightsquigarrow c=0$

?? $m\ddot{x}+cx=0$ $\qquad 1/c\neq 0 \rightsquigarrow 1/c=0$

- Ritz-type methods for impacts?

⇝ energy-preserving schemes