Variations on the periodic solutions of the forced pendulum

Jean Mawhin

Veteran of the Université Catholique de Louvain

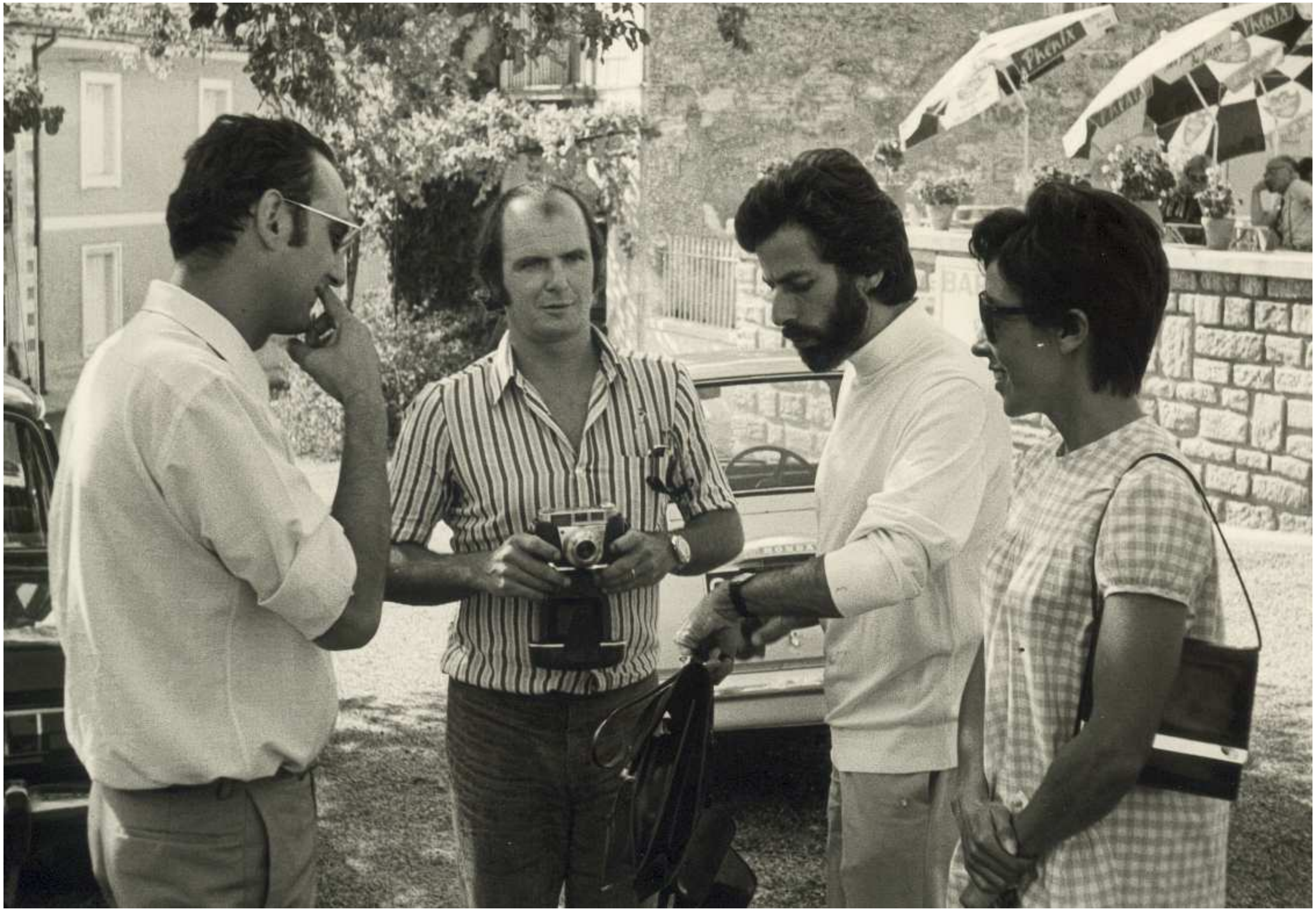
Dedication

- *Cordialement dédié à Michel Jean,*
  mon ami de plus de quarante ans,
  à l'occasion de son entrée
  dans sa deuxième décade de sexagénaire
- *attention : ce souhait n'est pas valable en Belgique !*

Souvenirs – 1

Souvenirs – 2

Souvenirs – 3



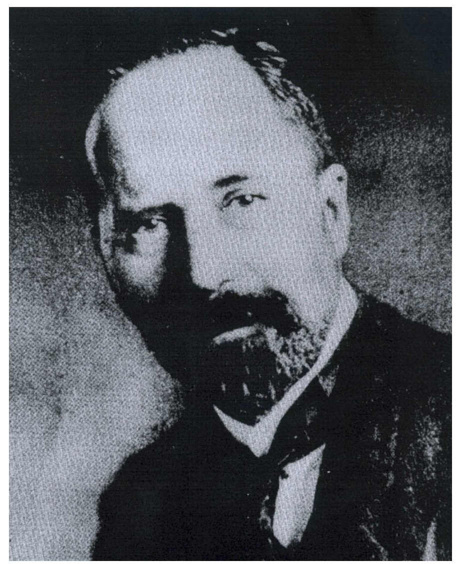
Souvenirs – 4

Souvenirs – 5

Souvenirs – 6

Souvenirs – 7

I. The classical forced pendulum equation



GEORG DUFFING (1861–1944)

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- p. 44 : $\psi'' + \frac{g}{L}\sin\psi = k\sin\omega t$
- first nonlinear approximation :
  $\psi'' + \frac{g}{L}\psi - \frac{g}{6L}\psi^3 = k\sin\omega t$

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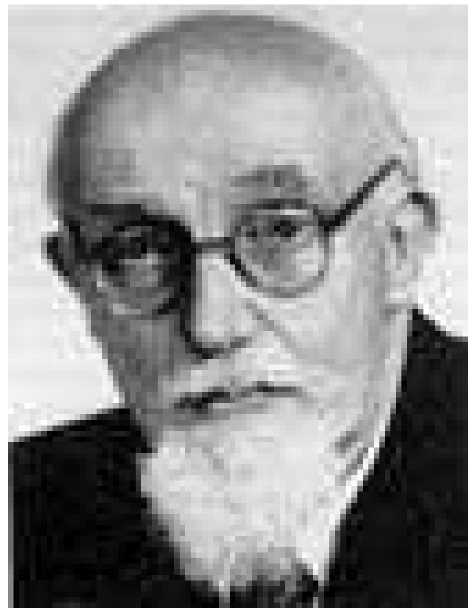
- first nonlinear approximation :
  $\psi'' + \frac{g}{L}\psi - \frac{g}{6L}\psi^3 = k\sin\omega t$
- T-periodic solutions for k small $T = \frac{2\pi}{\omega}$
- approximate amplitude vs frequency curve

From mechanics to analysis



GEORG HAMEL (1877-1954)

1922

- HAMEL (*Math. Annalen* **86**)
- $T = \frac{2\pi}{\omega}$
- $u'' + A\sin u = B\cos\omega t, \quad u(0) = u(T), \quad u'(0) = u'(T)$:
  Euler-Lagrange equation of the C^1 action functional
  $\mathcal{I}_H(u) = \int_0^T (\frac{u'^2}{2} + A\cos u + uB\cos\omega x)\,dt$
  on the space
  $C^1_\# := C^1_\#[0,T] = \{u \in C^1[0,T] \mid u(0) = u(T)\}$

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- $\mathcal{I}_H$ has a minimum over $C^1_\#$
- *at least one solution* $\forall\ A > 0, \quad \forall\ B \in \mathbb{R}$
- proof valid for $B \cos \omega t$ replaced by any $h(t) \in \widetilde{C^0}$

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- result is rapidly forgotten

Necessary conditions

$$u'' + A\sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

- $A > 0, \quad T > 0, \quad h \in L^1 := L^1(0, T)$
- existence and multiplicity of solutions

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- $A > 0, \quad T > 0, \quad h \in L^1 := L^1(0,T)$
- existence and multiplicity of solutions
- $S \subset L^1(0,T)$ vector subspace
  $\widetilde{S} := \left\{ v \in S \mid \overline{v} := \frac{1}{T}\int_0^T v(t)\,dt = 0 \right\}$
- NC for existence : $\overline{h} \in [-A, A]$
- NC for existence $\forall\, T > 0, \quad \forall\, A > 0 : \quad h \in \widetilde{L^1}$
- is $h \in \widetilde{L^1}$ sufficient ?

1980-82

- $W^{1,p}[0,T] := \{u \in L^p \,|\, u' \in L^p\} \quad (1 \leq p \leq +\infty)$
- $u'' + A\sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$:
  Euler-Lagrange equation of the C^1 action functional
  $\mathcal{I}(u) = \int_0^T (\frac{u'^2}{2} + A\cos u + uh)\, dt$
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  at least one solution $\quad \forall\, h \in \widetilde{L^2}, \quad \forall\, A < \omega^2$

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  independently : $\mathcal{I}$ *has a minimum over* $W^{1,2}_{\#}$
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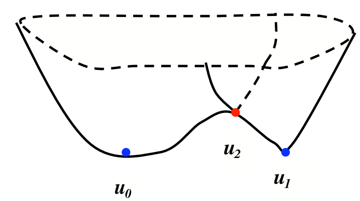
- none of them knew HAMEL's paper

1984

- M.–WILLEM (*J. Diff. Equ.* **52**) : $\forall\, h \in \widetilde{L^1}$
  - $\mathcal{I}$ *bounded below on* $W^{1,2}_{\#}$, $\mathcal{I}(u) = \mathcal{I}(u + 2\pi)$, $(PS)_c$ *and* (BPS)*-conditions* $\forall\, c \in \mathbb{R}$
  - $\mathcal{I}$ *has a minimum on* $W^{1,2}_{\#}$ *at* u_0
  - $\mathcal{I}$ *has the geometry of mountain pass lemma w.r.t.* u_0 *and* $u_0 + 2\pi$



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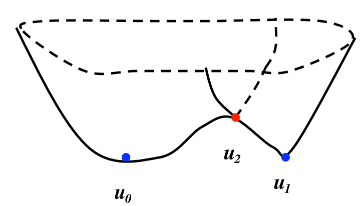

- $u'' + A \sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$ *has at least two solutions* $\forall\, A > 0, \quad \forall\, h \in \widetilde{L^1}$

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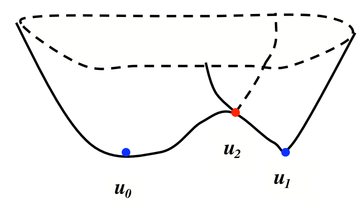

- $u'' + A \sin u = h(t)$, $u(0) = u(T)$, $u'(0) = u'(T)$ *has at least two solutions* $\forall\, A > 0$, $\forall\, h \in \widetilde{L^1}$
- sharp : $A \leq \omega^2$, $h \equiv 0$: $u_0 \equiv \pi$, $u_1 \equiv 0$

1988-89

- RABINOWITZ (*Trans. AMS* **310**), K.C. CHANG (*J. Nonlin. Anal.* **13**), M. (*Ann.Inst.H.Poincaré–ANL* **6** *suppl.*) *independently* : $\forall\, h \in \widetilde{L^1}$
  - $\mathcal{I}(u+2\pi) = \mathcal{I}(u)$ *can be considered on* $S^1 \times \widetilde{W^{1,2}_{\#}}$
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Remarks

- $A \sin u$ can be replaced by $g(t,u)$ periodic in u with $\int_0^{2\pi} g(t,u)\,du = 0$
- extensions to systems in $\mathbb{R}^N$: $u'' + \nabla_u F(t,u) = h(t)$
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- more informations and references in
  M., *Handbook Diff. Equ. ODE* **1** *(2004)*

II. The forced ‘p-pendulum equation’ ($p > 1$)

The problem

$$(|u'|^{p-2}u')' + A\sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

- $p > 1, \quad A > 0, \quad T > 0, \quad h \in L^1$
- solution : $\quad u \in C^1[0,T], \quad |u'|^{p-2}u' \in AC[0,T]$
- existence and multiplicity of solutions

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The action functional

- $(|u'|^{p-2}u')' + A\sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$
  Euler-Lagrange equation of action functional
  $\mathcal{I}_p(u) = \int_0^T (\frac{|u'|^p}{p} + A\cos u + uh)\, dt$

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  Euler-Lagrange equation of *action functional*
  $\mathcal{I}_p(u) = \int_0^T (\frac{|u'|^p}{p} + A\cos u + uh)\, dt$
- $\mathcal{I}_p$ is C^1 on
  $W_{\#}^{1,p} := \{u \in W^{1,p}[0,T] \mid u(0) = u(T)\}$
- $\forall\ h \in \widetilde{L^1} \quad \mathcal{I}_p(u + 2\pi) = \mathcal{I}_p(u)$
- $\forall\ h \in \widetilde{L^1} \quad \mathcal{I}_p$ *can be considered on* $S^1 \times \widetilde{W_{\#}^{1,p}}$

Multiplicity result

- M. *Le Matematiche, 2011*
- $\forall\ h \in \widetilde{L^1}$
  - $\mathcal{I}_p$ *bounded below, satisfies (PS)-condition on* $S^1 \times \widetilde{W^{1,p}_\#}$
  - $\mathcal{I}_p$ *has at least* $cat_{S^1\times\widetilde{W^{1,p}_\#}}(S^1 \times \widetilde{W^{1,p}_\#})$ *critical points* (LUSTERNIK-SCHNIREL'MAN-PALAIS)
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Remarks and open questions

- $A \sin u$ can be replaced by $g(t,u)$ periodic in u with $\int_0^{2\pi} g(t,u)\, du = 0$
- extensions to systems in $\mathbb{R}^N$
  $(\|u'\|^{p-2}u')' + \nabla_u F(x,u) = h(t)$
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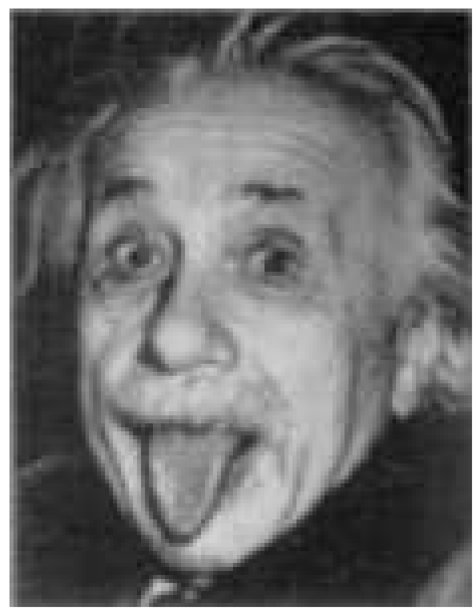
- existence and multiplicity results using fixed point theory and degree under some conditions upon A and T (BEREANU-JEBELEAN-M. *J.Dyn.Diff.Equ. (2010)*)
- replace the p-Laplacian $(|u'|^{p-2}u')'$ by $(\varphi(u'))'$ for a suitable class of homeomorphisms $\varphi : \mathbb{R} \to \mathbb{R}$ (Orlicz spaces) ?

III. The forced ‘relativistic pendulum equation’



The problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + A\sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

- $A > 0, \quad T > 0, \quad h \in L^1$
- solution : $u \in C^1[0,T], \quad \|u'\|_\infty < 1,$ $\frac{u'}{\sqrt{1-u'^2}} \in AC[0,T]$

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- BREZIS–M., *Diff.Int.Equ.* **23** *(2010)*

The action functional

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  $= \{u : [0,T] \to \mathbb{R} \mid u \; Lipschitzian, \; u(0) = u(T)\}$
- $W^{1,\infty}_{\#}$ *Banach space with norm*
  $\|u\|_{1,\infty} := \|u\|_\infty + \|u'\|_\infty$

The action functional

- $\mathcal{I}_r(u) = \int_0^T (-\sqrt{1-u'^2} + A\cos u + uh)\,dt$
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- $W_{\#}^{1,\infty}$ *Banach space with norm*
  $\|u\|_{1,\infty} := \|u\|_\infty + \|u'\|_\infty$
- $\mathcal{I}_r$ *defined on the closed convex set*
  $K = \{u \in W_{\#}^{1,\infty} \mid \|u'\|_\infty \leq 1\}$
  of class C^1 *on* $\{u \in W_{\#}^{1,\infty} \mid \|u'\|_\infty < 1\}$

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  not a priori Euler-Lagrange equation of $\mathcal{I}_r$
- finding a critical point of $\mathcal{I}_r$ not sufficient to have a T-periodic solution

Minimization of $\mathcal{I}_r$

- $\forall\ (u_j)$ *in* K *converging in* $C^0[0,T]$ *to* $u \in K$, $\liminf_{j\to\infty} \int_0^T (-\sqrt{1-u_j'^2}\,)\,dt \geq \int_0^T (-\sqrt{1-u'^2}\,)\,dt$

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- $\forall\, h \in \widetilde{L^1}$, $\mathcal{I}_r$ *has a minimum over* K
  - equivalent to minimize $\mathcal{I}_r$ on $\widehat{K} := \{u \in W^{1,1}_{\#} \mid \overline{u} \in [0, 2\pi],\ \|u'\|_\infty \leq 1\}$
  - $\widehat{K}$ bounded and equicontinuous
  - up to subsequence any minimizing sequence converges uniformly to some $u^* \in K$
  - u^* minimizes $\mathcal{I}_r$ on K

Variational inequality

- $\mathcal{I}_r(u) = \min_K \mathcal{I}_r \quad \Rightarrow \quad \forall\, v \in K$
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- problem : show that $\|u'\|_\infty < 1$

Auxiliary problem

- $\forall\ f \in L^1$
  $$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' - u = f(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$
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Existence result

- $\forall\ h \in \widetilde{L^1}, \quad \forall\ A > 0$ *problem*
  $$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + A\sin u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$
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- idea of proof. $f_w := -A\sin w + h - w \in L^1 \quad (w \in K)$
  - u solves the v.i. : $\forall\, v \in K$
    $$\int_0^T \left[-\sqrt{1-v'^2} + \sqrt{1-u'^2} + (u + f_u)(v-u)\right]\, dt \geq 0$$
  - $\forall\, w \in K$ the unique T-periodic solution $\widehat{u}_{f_w}$ of
    $\left(\frac{u'}{\sqrt{1-u'2}}\right)' - u = f_w$ satisfies the v.i. $\forall\, v \in K$
    $$\int_0^T \left[-\sqrt{1-v'^2} + \sqrt{1-\widehat{u}'^2_{f_w}} + (\widehat{u}_{f_w} + f_w)(v - \widehat{u}_{f_w})\right]\, dt \geq 0$$
  - this implies $u = \widehat{u}_{f_u}$ and hence $\|u'\|_\infty < 1$

Remarks and open questions

- $A \sin u$ can be replaced by $g(t, u)$ periodic in u with $\int_0^{2\pi} g(t, u)\, du = 0$
- $-\sqrt{1 - u'^2}$ can be replaced by $\Phi(u')$ with $\Phi \in C[-a, a] \cap C^1(-a, a)$ strictly convex $\phi = \Phi' : (-a, a) \to \mathbb{R}$ homeomorphism with $\phi(0) = 0$

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- existence and multiplicity results by degree theory for $\overline{h}$ possibly non-zero, under restrictions upon A and T using fixed point theory (Torres *Com.Cont.Math.*, Bereanu-Jebelean-M. *J.Dyn.Diff.Equ. (2010)*)

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- open problem : multiplicity of solutions

hypnosis under the pendulum over !
Michel, please wake up !